Deep Inference for Graphical Theorem Proving

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PhD defense, Palaiseau

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Introduction

Context

Logic

- Study of *sound* reasoning

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- Example of everyday life deduction:

$$\frac{\text{it rains 🌧️} \qquad \text{you don’t have ☂️}}{\text{you are wet 💧}}$$

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premisses	it rains 🌧️ you don't have ☂️
conclusion	you are wet 💧

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- Example of everyday life deduction:

premisses

conclusion

it rains 🌧️ you don't have ☂️ you are outside
you are not under a bus shelter ...

you are wet 💧

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- Example of everyday life deduction:

$$\frac{\text{it rains 🌧️} \qquad \text{you don't have ☂️} \qquad \text{you are outside} \qquad \text{you are not under a bus shelter} \qquad \ldots}{\text{you are wet 💧}} \quad \begin{matrix}\textbf{premisses}\\ \textbf{conclusion}\end{matrix}$$

- Hidden assumptions ⇒ lack of **certainty**

Formal logic

Let's try another one (Aristotle – 4th century BC):

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↳ Formal essence of logical reasoning

Mathematical logic

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$$\frac{x \text{ is } P \qquad \text{All } P \text{ are } Q}{x \text{ is } Q} \quad \rightsquigarrow \quad \frac{P(x) \qquad \forall y. P(y) \Rightarrow Q(y)}{Q(x)}$$

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- *Generic patterns* of deduction as **rules**

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Maths as a huge **game**

Goal: to *prove theorems* by following *inference rules*

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Goal: to *prove theorems* by following *inference rules*

- **Proof theory:** design & study of *rule systems* capturing maths

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- Problem: maths is *hard* ⇒ need for a **human** in the loop

↳ Interactive Theorem Provers (ITPs)

Contributions

Textual vs. Graphical

State-of-the art: build proofs by writing **textual** commands

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🙇 : “Please apply *this* rule”

💻 : “✅ OK here is the result!”

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💻 : “❌ ERROR: dkfsljfjdklsfjdkfjsldjfkdlsfj”

Textual vs. Graphical

State-of-the art: build proofs by writing **textual** commands

🙇 : “Please apply *this* rule”

💻 : “❌ ERROR: dkfsljfjdklsfjdkfjsldjfkdlsfj”

1st contribution: build proofs by direct manipulation of *formulas*

↳ No need to *memorize* the rules

↳ More *straightforward* interaction

Symbolic vs. Iconic

- **Symbols** are hard to:
  - *learn* ⇒ purely conventional meaning
  - *manipulate* ⇒ need for very precise gestures

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Symbolic vs. Iconic

- **Symbols** are hard to:
  - *learn* $\Rightarrow$ purely conventional meaning
  - *manipulate* $\Rightarrow$ need for very precise gestures
- Formulas can **interact** by being *moved* in the same space

2nd contribution: replace logical symbols by geometrical diagrams

Symbolic Manipulations

Proof-by-Action

*A **demo** is worth a thousand words!*

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- Fully graphical: no textual proof language

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- Both spatial and temporal:

proof = gesture sequence

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- Both spatial and temporal:

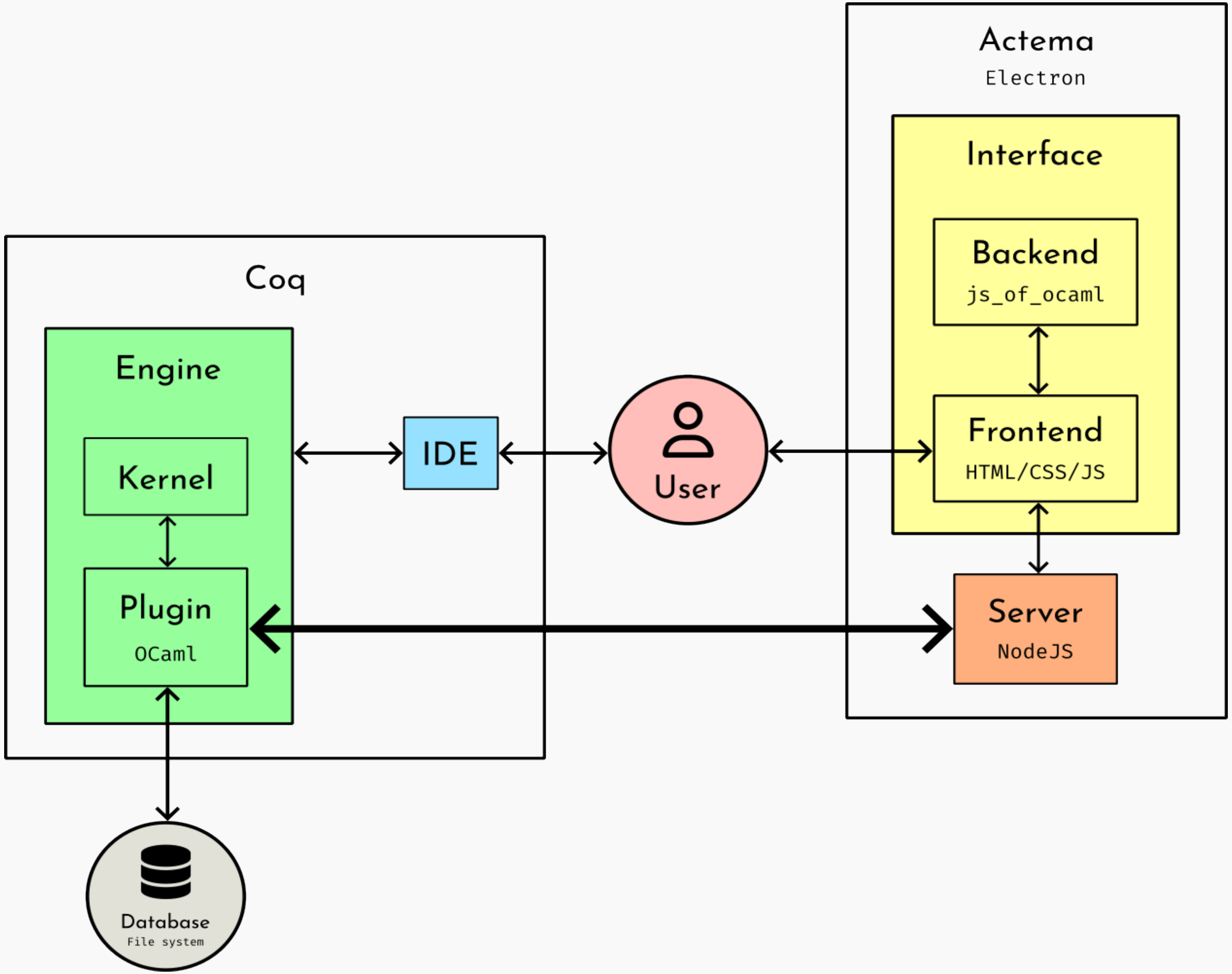
$$\text{proof} \quad = \quad \text{gesture sequence}$$

- **Different modes** of reasoning with a **single "syntax"**:

$$\begin{array}{rcl} \text{Click} & \iff & \text{introduction/elimination} \\ \text{Drag-and-Drop} & \iff & \text{backward/forward} \end{array}$$

coq-actema

(Bouverot, Donato, Najjar, Strub, Werner)



Semantics of Drag-and-Drop

Subformula Linking (Chaudhuri 2013)

Idea: bring matching subformulas through switch rules

$$\underline{A} \wedge B \circledcirc B \wedge (\underline{A} \vee C) \wedge D$$

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$$\text{switch}\left\{\begin{array}{rl} & \underline{A} \wedge B \mathbin{⧁} B \wedge (\underline{A} \vee C) \wedge D \\ \to & B \wedge (\underline{A} \wedge B \mathbin{⧁} (\underline{A} \vee C) \wedge D) \\ \to & B \wedge (\underline{A} \wedge B \mathbin{⧁} \underline{A} \vee C) \wedge D \\ \to & B \wedge ((\underline{A} \wedge B \mathbin{⧁} \underline{A}) \vee C) \wedge D \\ \to & B \wedge ((B \Rightarrow \underline{A} \mathbin{⧁} A) \vee C) \wedge D \end{array}\right.$$

Subformula Linking (Chaudhuri 2013)

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$$
\text{switch}\left\{
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\to & B \wedge ((\underline{A} \wedge B \mathbin{⧁} \underline{A}) \vee C) \wedge D \\
\to & B \wedge ((B \Rightarrow \underline{A} \mathbin{⧁} \underline{A}) \vee C) \wedge D
\end{array}
\right.
$$

$$
\text{identity}\left\{ \to B \wedge ((B \Rightarrow \top) \vee C) \wedge D \right.
$$

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\quad \underline{A} \wedge B \ogreaterthan B \wedge (\underline{A} \vee C) \wedge D \\
\to B \wedge (\underline{A} \wedge B \ogreaterthan (\underline{A} \vee C) \wedge D) \\
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$$

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Variant of the **Calculus of Structures** (Guglielmi 1999)

Linking under quantifiers (Donato, Strub, and Werner 2022)

$\exists y.\ \forall x.\underline{R(x,y)} \otimes \forall a.\exists b.\underline{R(a,b)}$

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- Unify linked subformulas

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$$x \longmapsto a$$

$$y \longleftarrow\!\shortmid\ b$$

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- **Check** for ∀∃ dependency cycles

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✓

- Unify linked subformulas
- **Check** for ∀∃ dependency cycles
- **Switch** uninstantiated quantifiers

$$\exists y. \forall x. \underline{R(x,y)} \otimes \forall a. \exists b. \underline{R(a,b)}$$

$$\to \forall y. \left(\forall x. \underline{R(x,y)} \otimes \forall a. \exists b. \underline{R(a,b)} \right)$$

$$\to \forall y. \forall a. \left(\forall x. \underline{R(x,y)} \otimes \exists b. \underline{R(a,b)} \right)$$

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✓

Linking under quantifiers (Donato, Strub, and Werner 2022)

- Unify linked subformulas
- Check for $\forall\exists$ dependency cycles
- Switch uninstantiated quantifiers
- Instantiate unified variables

$$\exists y. \forall x. R(x, y) \ogreaterthan \forall a. \exists b. R(a, b)$$

$$\to \forall y. \big(\forall x. R(x, y) \ogreaterthan \forall a. \exists b. R(a, b) \big)$$

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$$\to \forall y. \forall a. \big(\forall x. R(x, y) \ogreaterthan R(a, y) \big)$$

$$\to \forall y. \forall a. (R(a, y) \ogreaterthan R(a, y))$$

$$\to^* \top$$

$$x \longmapsto a$$

$$y \longmapsfrom b$$

$\checkmark$

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$$x \longleftarrow\!\!\dashv a$$

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- **Check** for ∀∃ dependency cycles

$$\forall a.\exists b.\underline{R(a,b)} \ogreaterthan \exists y.\forall x.\underline{R(x,y)}$$

$$\begin{tikzcd} x \arrow[d, bend right] & a \arrow[l, mapsto] \\ y \arrow[r, mapsto] & b \arrow[u, bend right] \end{tikzcd}$$

✗

Rewriting equalities (Donato, Strub, and Werner 2022)

Add 4 rules $\Longrightarrow$ `rewrite` tactic for free!

$$\underline{t} = u \otimes A \quad \to \quad A[u/t] \qquad\qquad t = \underline{u} \otimes A \quad \to \quad A[t/u]$$

$$\underline{t} = u \circledast A \quad \to \quad A[u/t] \qquad\qquad t = \underline{u} \circledast A \quad \to \quad A[t/u]$$

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Compositional with semantics of **connectives**:

- **Quantifiers:** rewrite modulo *unification*
- **Implication:** *conditional* rewrite
- **Arbitrary** combinations are possible:

$$\forall x. x \neq 0 \Rightarrow \underline{f(x)} = g(x) \circledcirc \exists y. A\Big(\underline{f(y)}\Big) \vee B(y)$$

$$\to^{*} \ \exists y. (y \neq 0 \wedge A(g(y))) \vee B(y)$$

Completeness

Add the following rules:

- Init $\quad C^+\boxed{A \Rightarrow B} \to C^+\boxed{A \circledgt B} \qquad C^-\boxed{A \wedge B} \to C^-\boxed{A \circledast B}$
- Release $\quad C^+\boxed{A \circledgt B} \to C^+\boxed{A \Rightarrow B} \qquad C^-\boxed{A \circledast B} \to C^-\boxed{A \wedge B}$
- Contraction $\quad C^-\boxed{A} \to C^-\boxed{A \wedge A}$

Theorem (Completeness): If $\Gamma \vdash A$ is provable in sequent calculus, then

$$\bigwedge \Gamma \Rightarrow A \to^* \top$$

Conclusion

- `coq-actema` still in development, but *already usable*
  ↳ follow install instructions on GitHub!
- Based on the solid proof theory of **subformula linking**
- Next step: exposure to *real users*
  - **Beginners/students:** introductory logic/proof assistants course
  - **Experts:** real maths codebases

Iconic Manipulations

Bubble Calculi

The chemical metaphor

Item	$\Longleftrightarrow$	Ion
Color	$\Longleftrightarrow$	Polarity
Logical connective	$\Longleftrightarrow$	Chemical bond
Click	$\Longleftrightarrow$	Heating
Drag-and-Drop	$\Longleftrightarrow$	Bimolecular reaction

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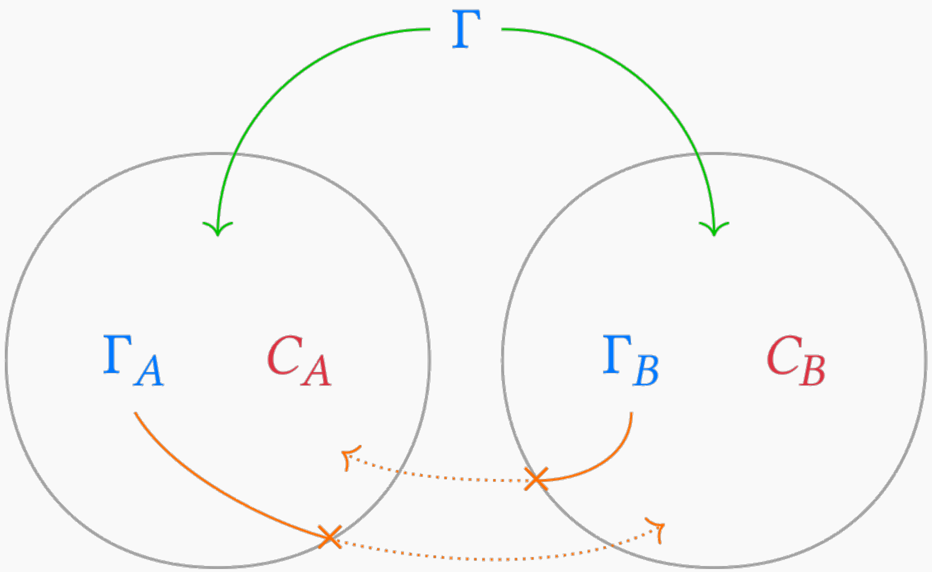
Breaks on rules that create subgoals (e.g. click on $\wedge$)

Bubbles

Natural way to depict context scoping



Two main inspirations:

- The **chemical abstract machine** (Berry and Boudol 1989)
- **Nested sequents** (Brünnler 2009)

Example proof

$$(A \vee B \Rightarrow C) \Rightarrow (A \Rightarrow C) \wedge (B \Rightarrow C)$$

Example proof

$A \vee B \Rightarrow C \qquad (A \Rightarrow C) \wedge (B \Rightarrow C)$

Example proof

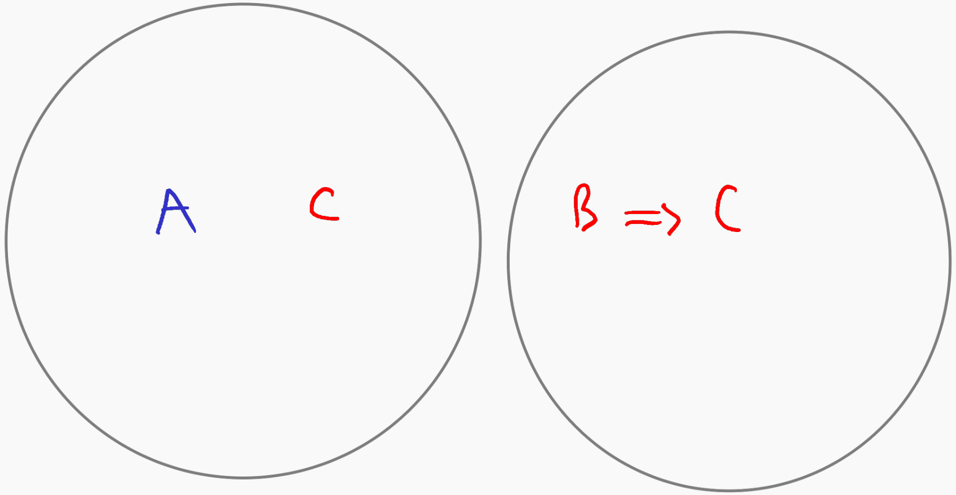
$A \vee B \Rightarrow C$

$A \Rightarrow C$

$B \Rightarrow C$

Example proof

$A \vee B \Rightarrow C$



Example proof

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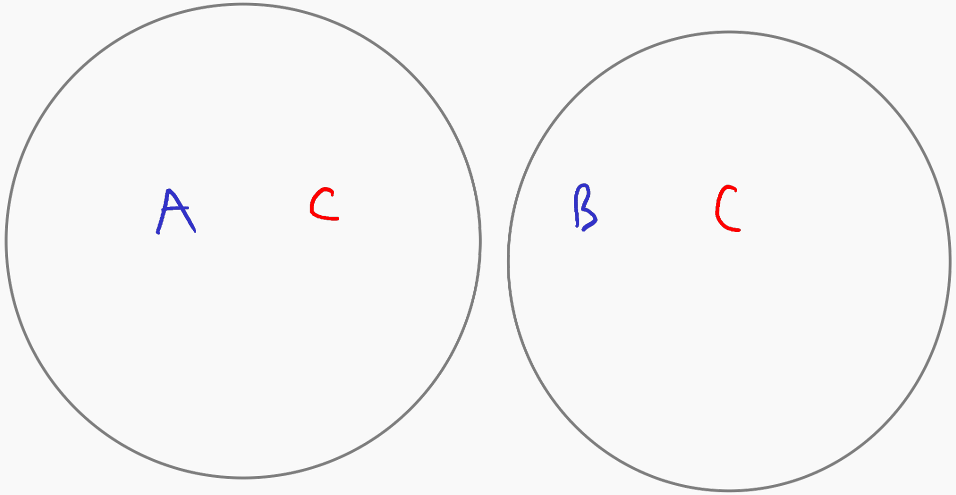

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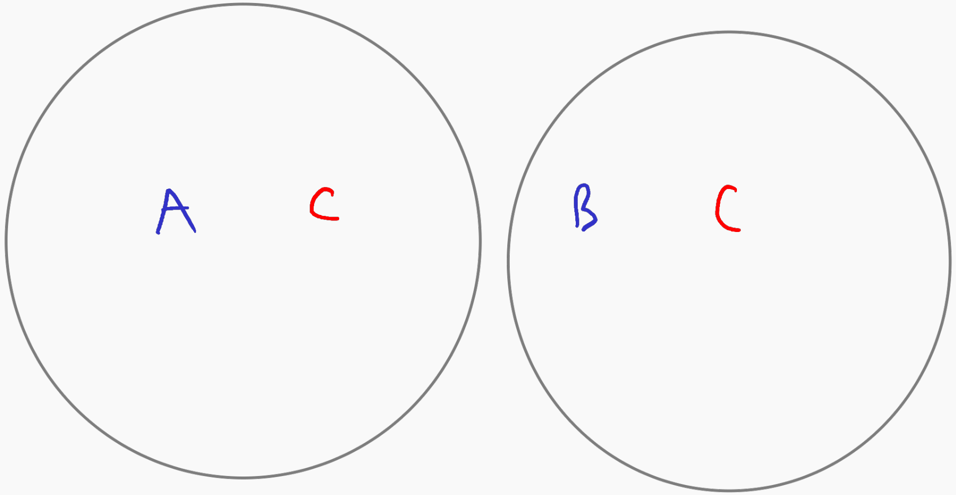

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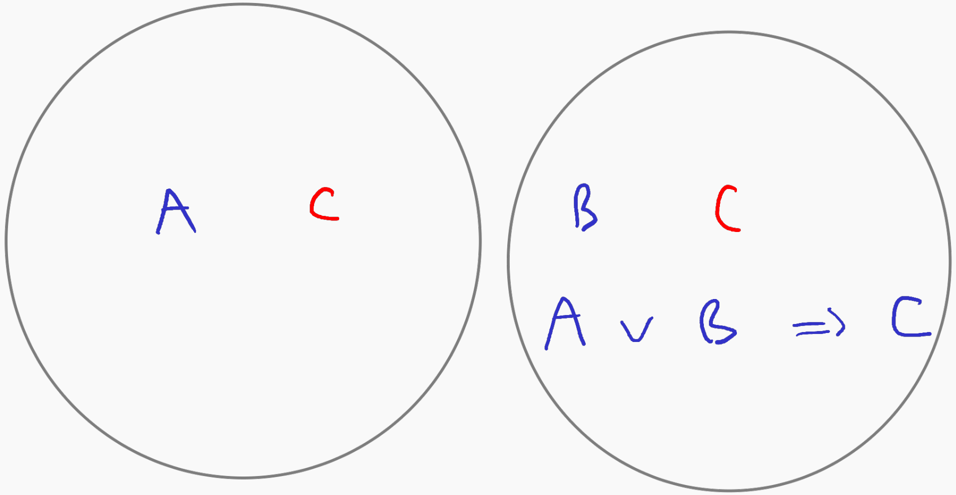

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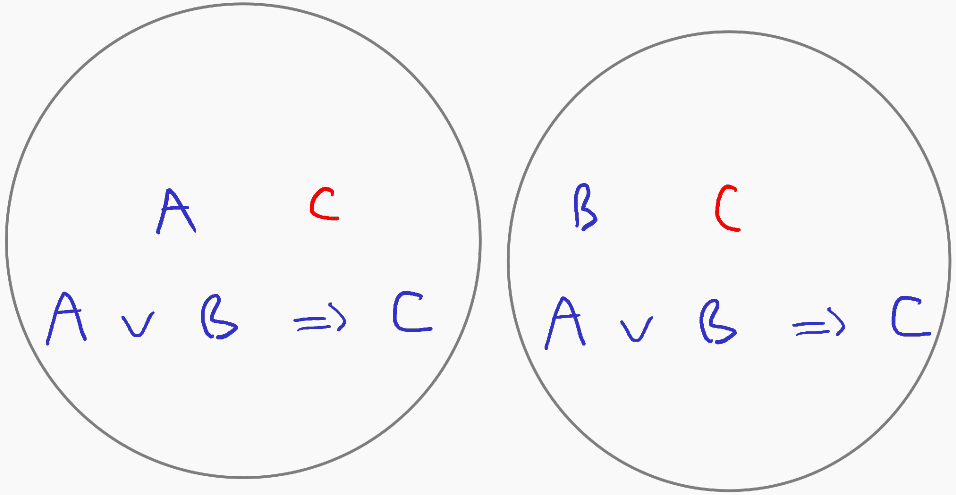

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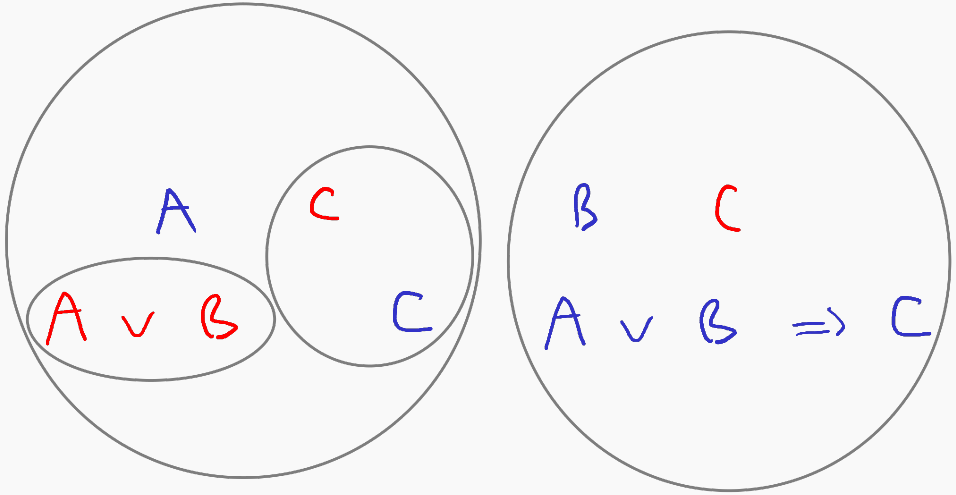

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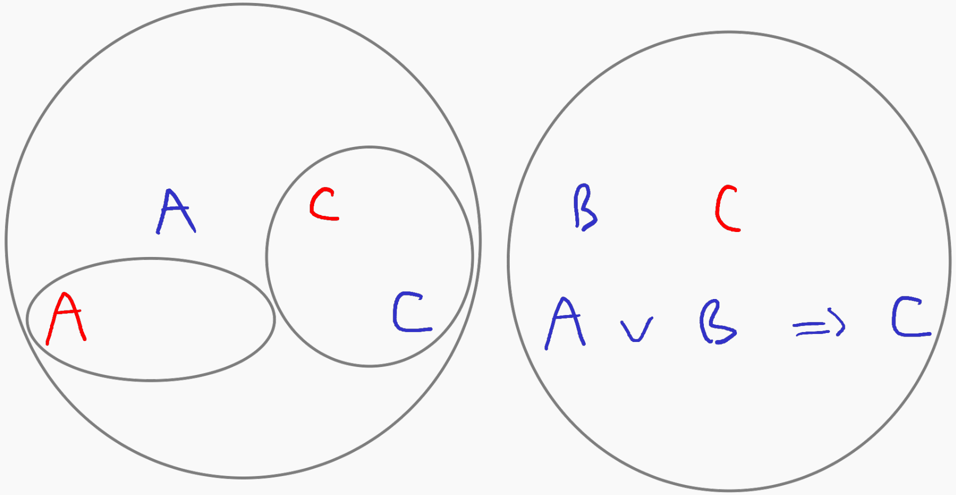

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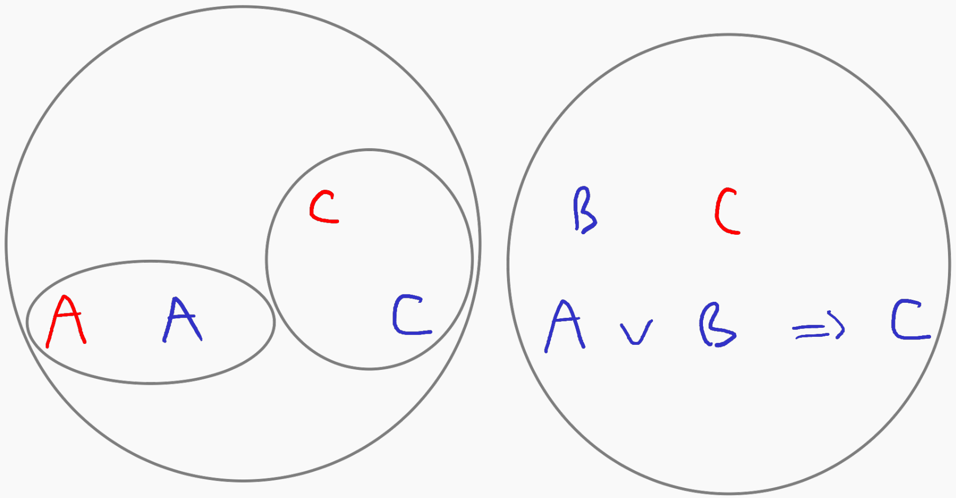

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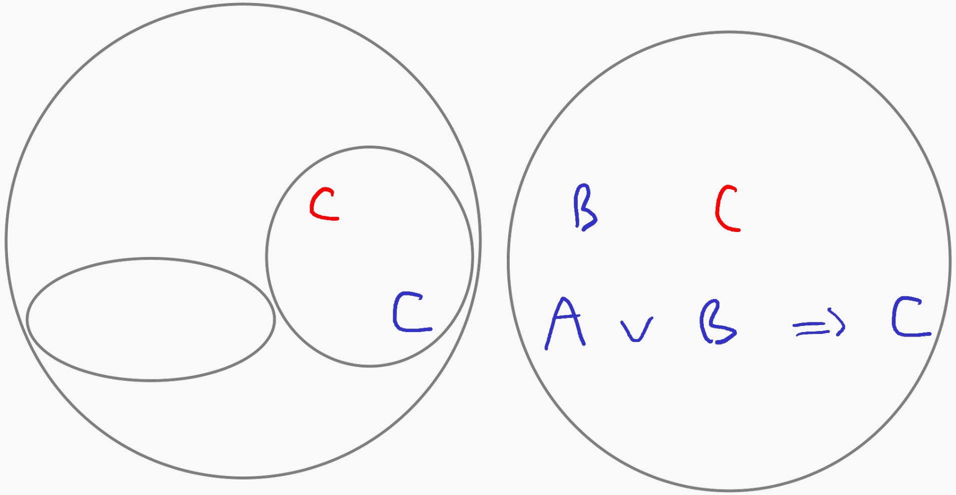

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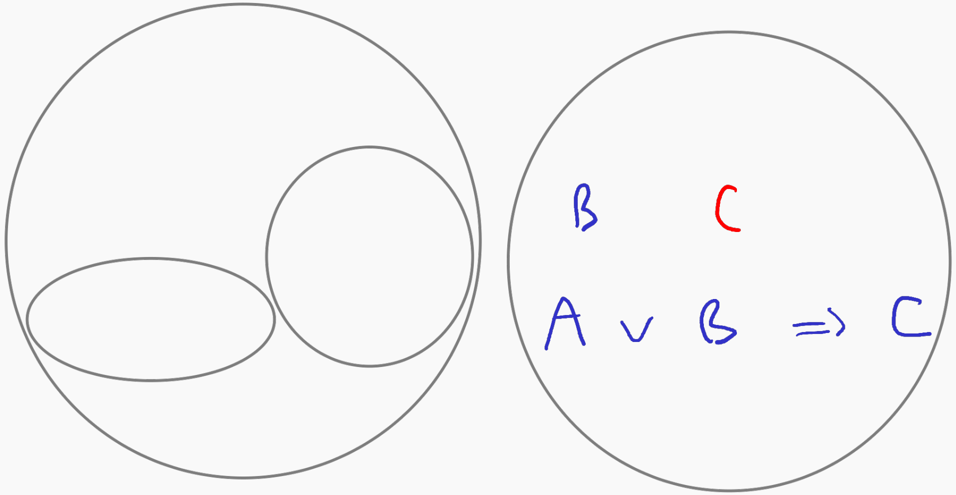

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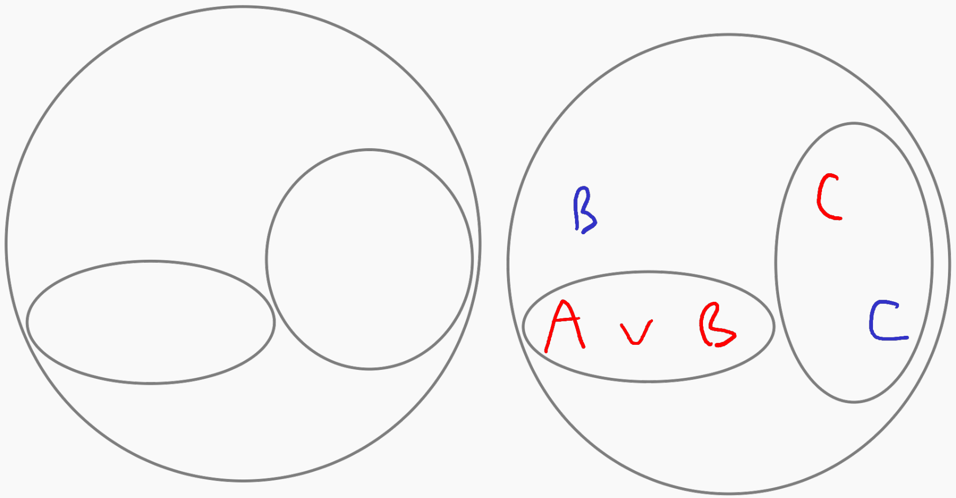

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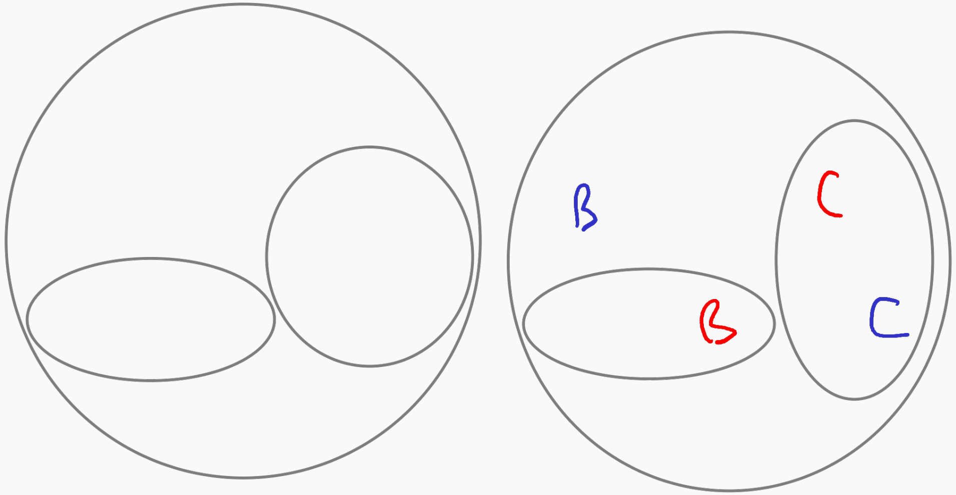

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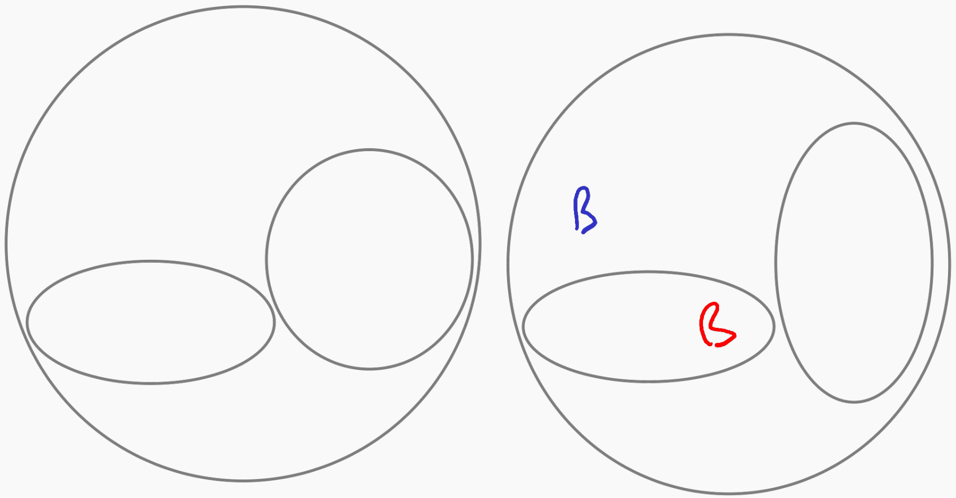

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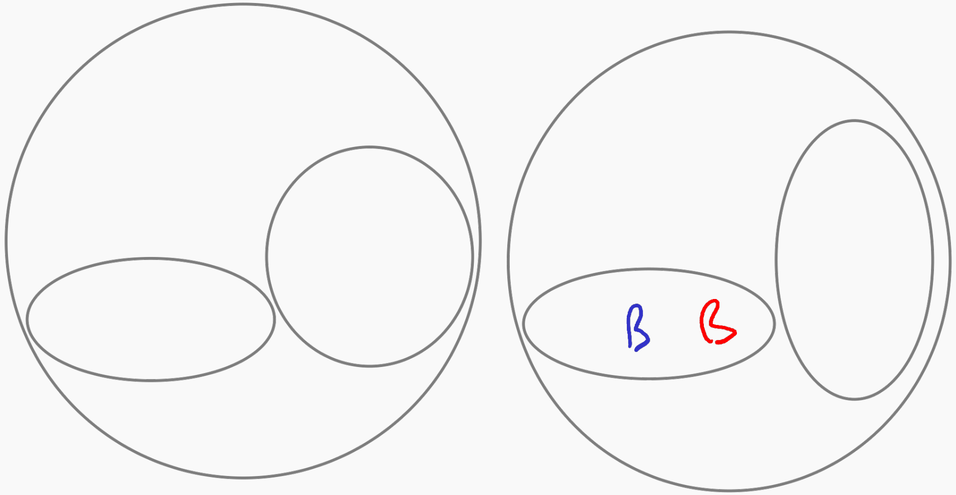

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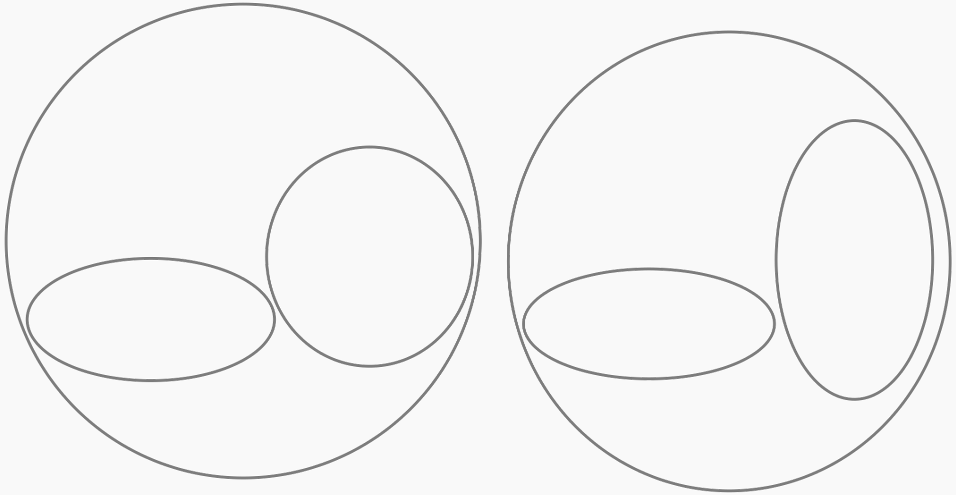
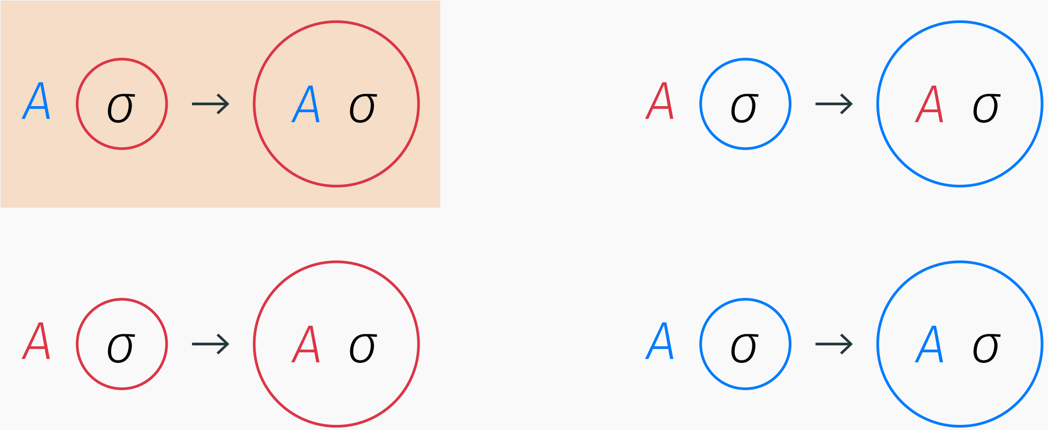


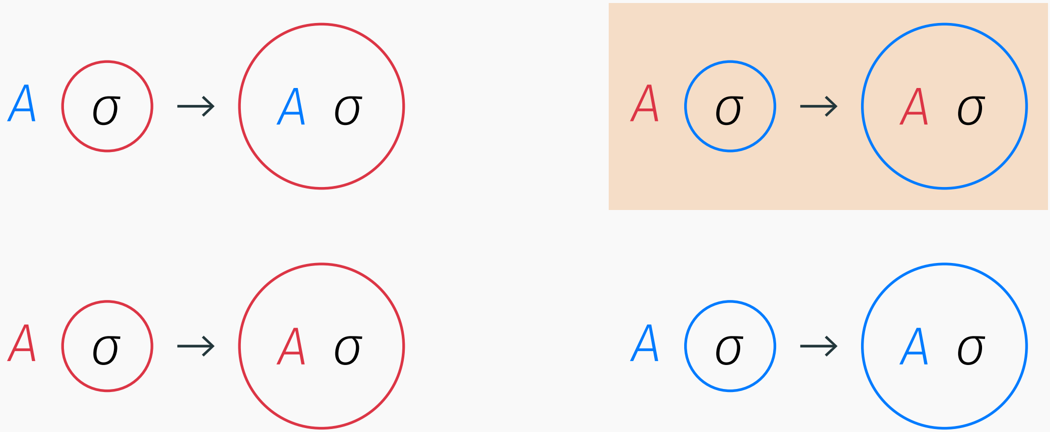
Example proof

Polarized bubbles



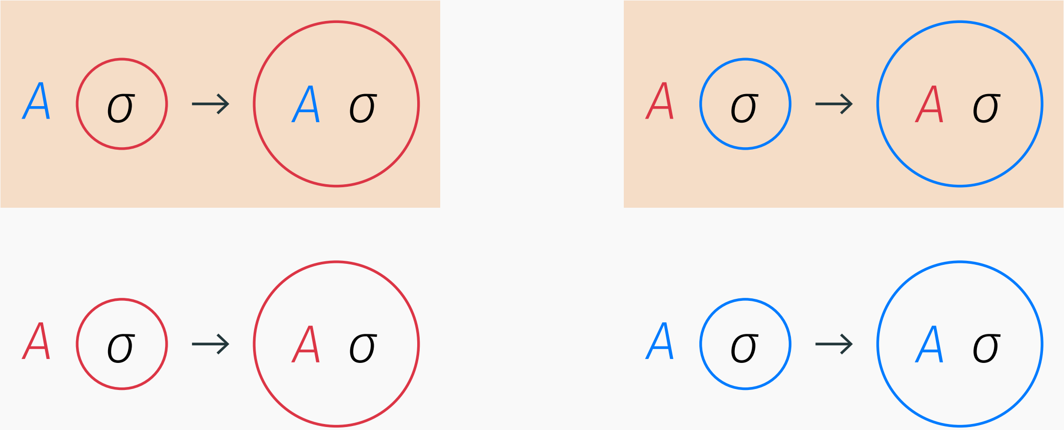
Intuitionistic logic

Polarized bubbles



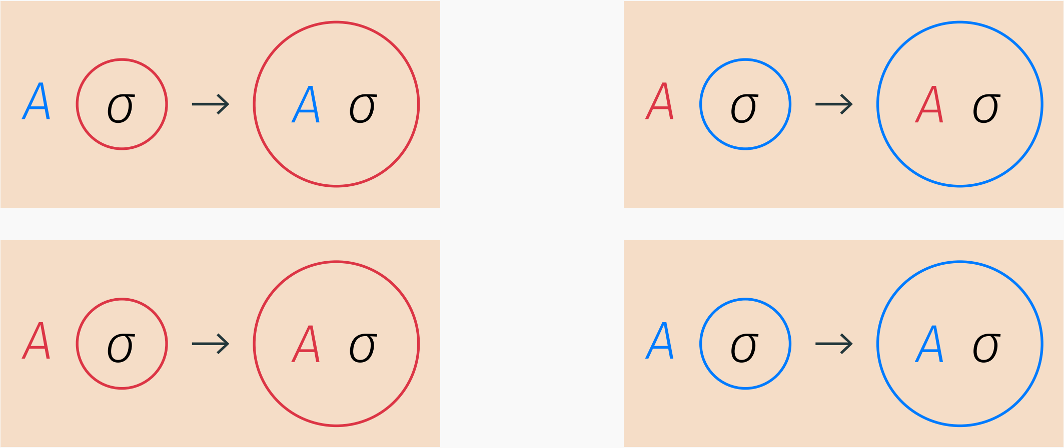
Dual-intuitionistic logic

Polarized bubbles



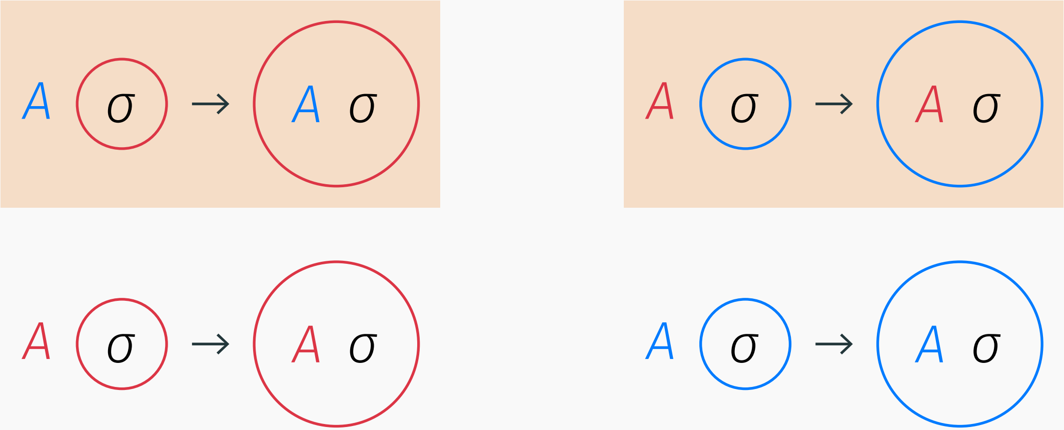
Bi-intuitionistic logic

Polarized bubbles



Classical logic

Polarized bubbles



Intuitionism = same polarities repel eachother

Flower Calculus

Polarity meets Space

Bubble calculi are not **fully iconic** (need for *symbolic* connectives)

Polarity meets Space

Bubble calculi are not **fully iconic** (need for *symbolic* connectives)

Key insight: **space** is *polarized*, not **objects**

Existential Graphs (Peirce 1906)

- **Diagrammatic** proof system invented by C. S. Peirce around 1890
- Topological representation of **negation** as nested "*cuts*" (Jordan curves):

$A \quad B$	A
$A \wedge B$	$\neg A$

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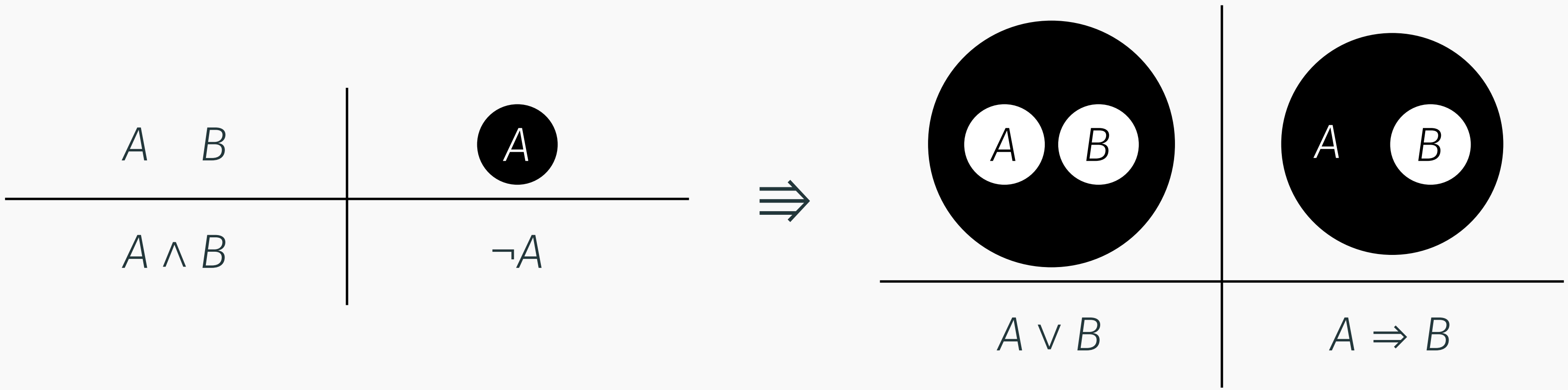
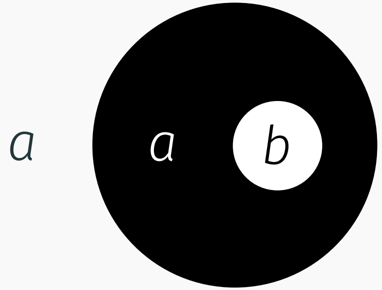


Inference rules on locations

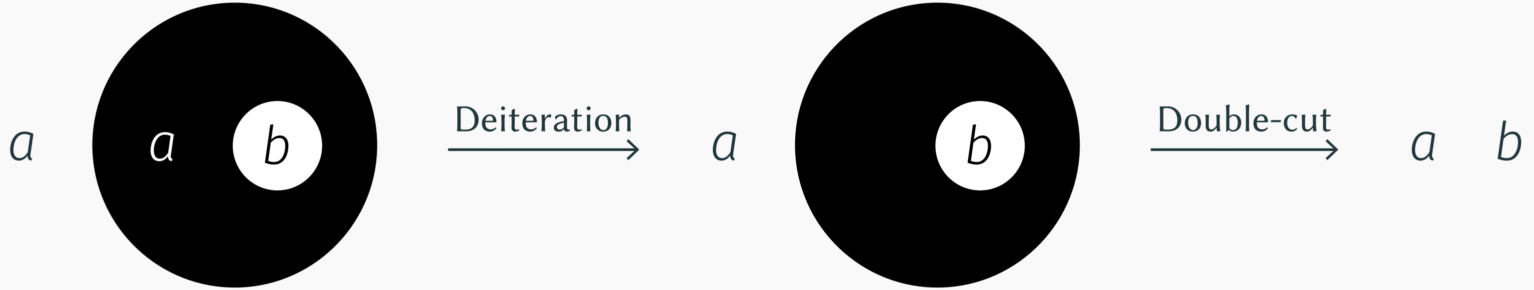


Illative transformations

Inference rules on locations

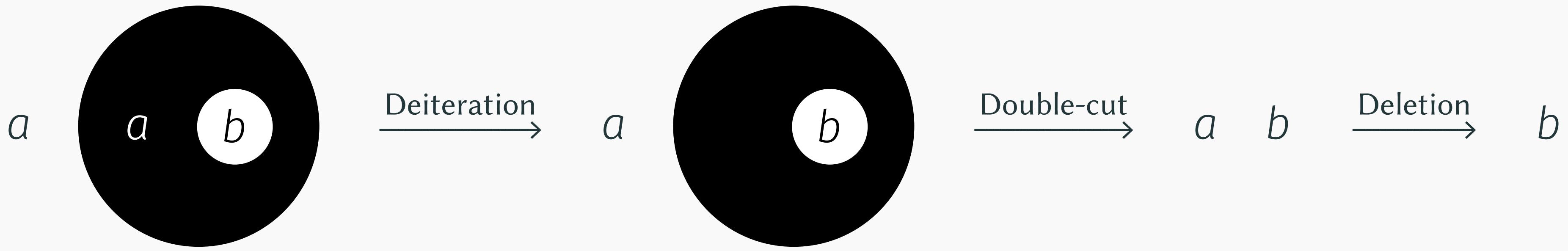
Illative transformations

Inference rules on locations

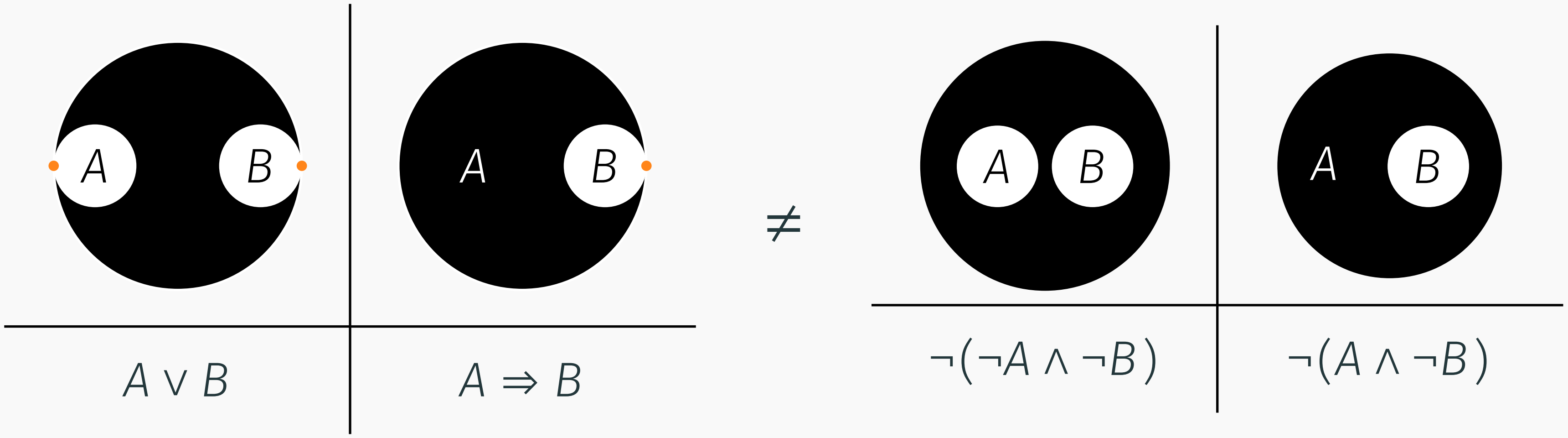


Illative transformations

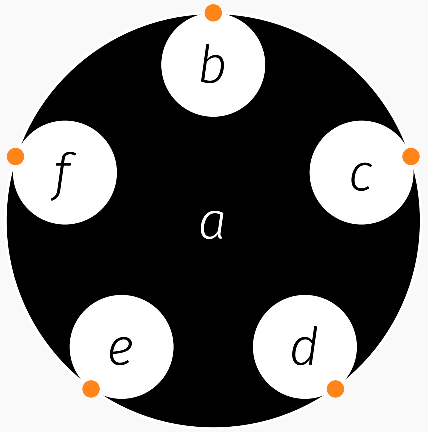
Inference rules on locations



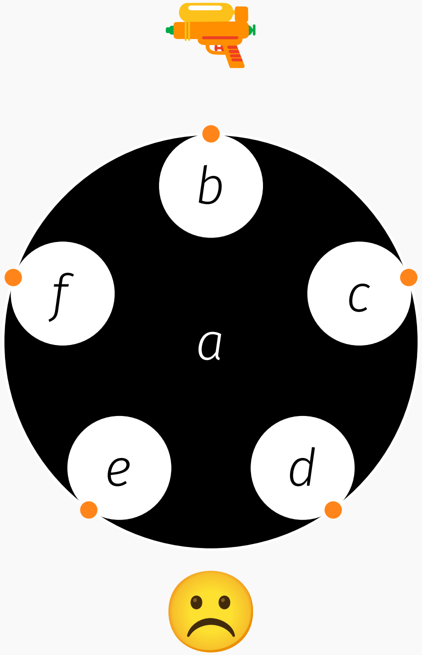
- Topological representation of **implication** with Peirce's "*scroll*"
- Scroll = continuously joined nested cuts:



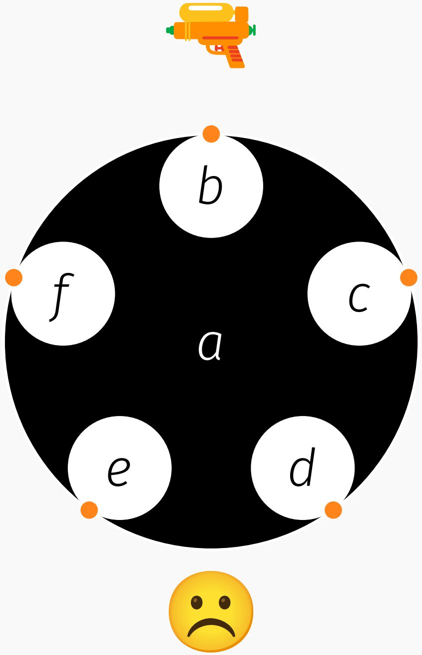
Blooming

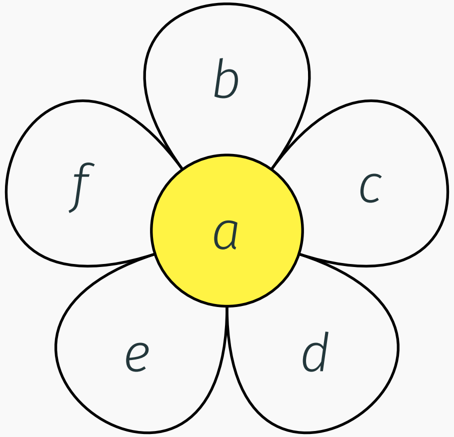


Blooming



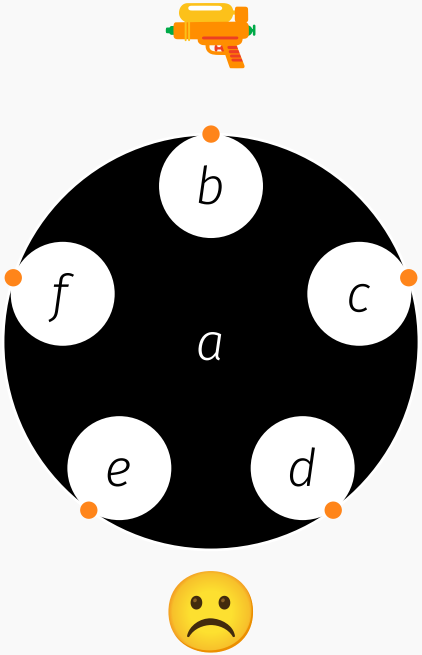
Blooming

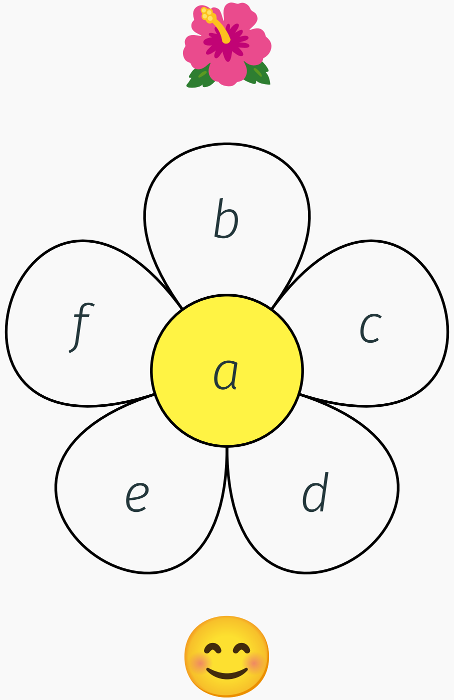




Turn **inloops** into petals.

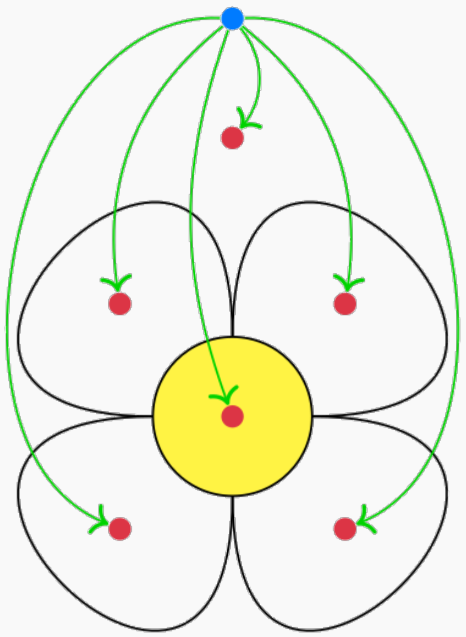
Blooming



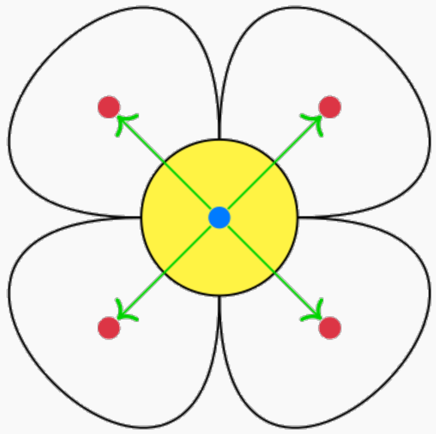


"Make love, not war"

Pollination



Cross-pollination



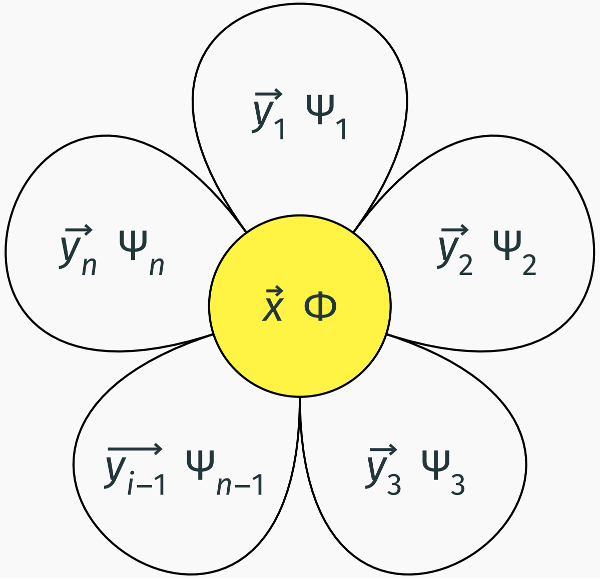
Self-pollination

Flower Calculus

(Donato 2024)

- Support for **quantification** with *binders* $\vec{x}$
- Interpretation as geometric formulas from *topos theory*
- Inference rules divided in two fragments:
  - **Nature** ❀ = analytic and invertible
  - **Culture** ✂ = non-invertible

Theorem (Analytic completeness): If a flower is *valid* (i.e. true in every Kripke model), then it is ❀-provable.



$$\forall \vec{x}.\left(\bigwedge \Phi \Rightarrow \bigvee_i \exists \vec{y}_i.\Psi_i \right)$$

Flower Prover

(mascarpone cream)

GUI in the Proof-by-Action paradigm based on the flower calculus

- Represent flowers as nested **boxes**
- **Modal interface** to interpret gestural actions:

Proof mode $\Longleftrightarrow$ Natural (invertible and analytic) rules

Edit mode $\Longleftrightarrow$ Cultural (non-invertible) rules

Navigation mode $\Longleftrightarrow$ Contextual closure (functoriality)

Thank you!

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