The Flower Calculus

An interactive and diagrammatic approach to constructive proofs

Pablo Donato (Grothendieck Institute)

2025-01-10

Joint Mathematics Meetings 2025

AMS Special Session on Methods of Compassionate Math II

Interactive theorem proving

- **Dream:** a powerful software environment for *formal mathematics*

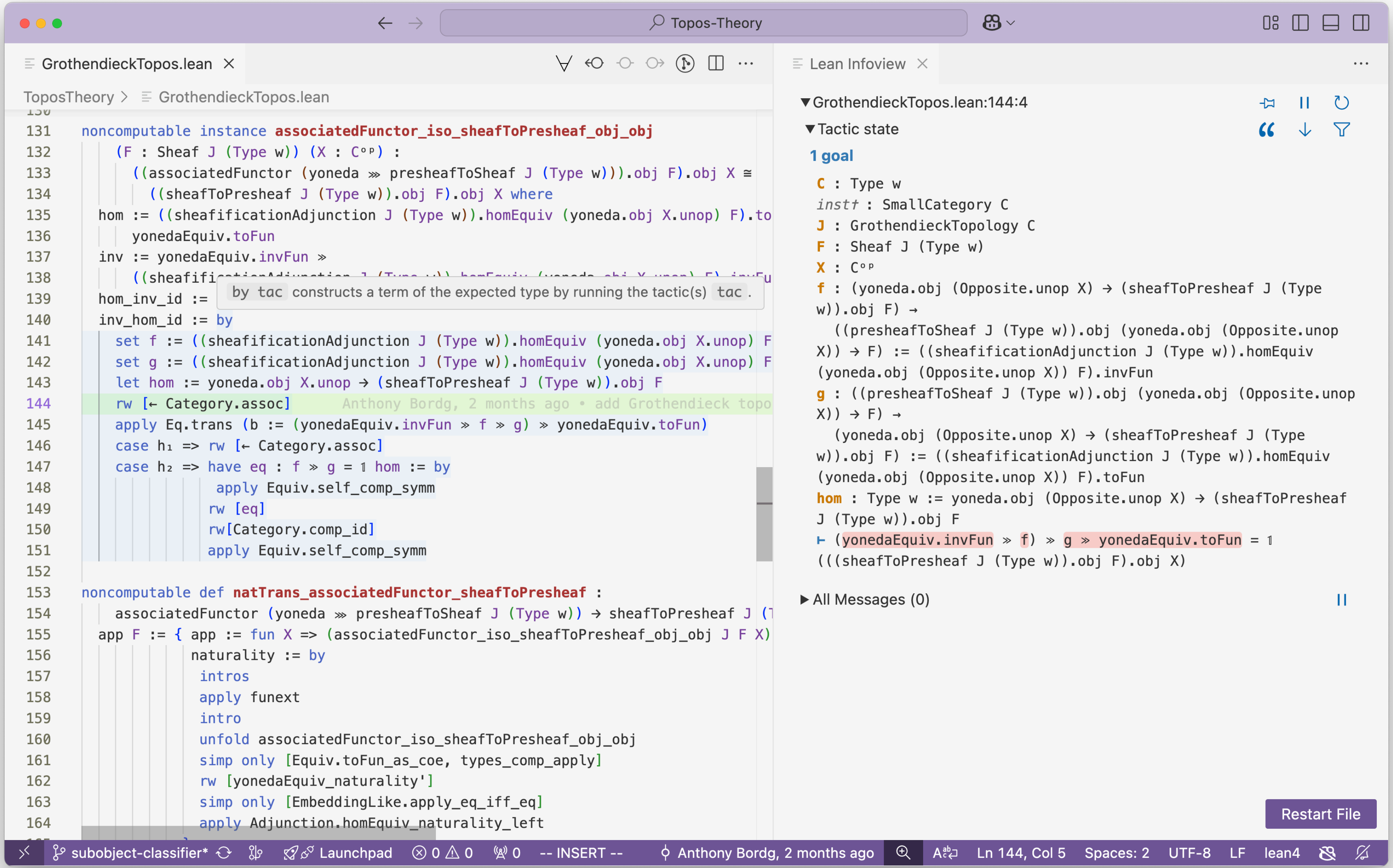
Interactive theorem proving

- **Dream:** a user-friendly, yet powerful software environment for *formal mathematics*

Interactive theorem proving

- **Dream:** a user-friendly, yet powerful software environment for *formal mathematics*
- **Problem**: current interfaces (and incoming ones based on LLMs) are mostly textual and verbal

$$\underbrace{\text{Commands}}_{\text{Proofs}} \quad \text{on} \quad \underbrace{\text{Symbolic expressions}}_{\text{Statements}}$$


Topos-Theory
GrothendieckTopos.lean
ToposTheory > GrothendieckTopos.lean
noncomputable instance associatedFunctor_iso_sheafToPresheaf_obj_obj
(F : Sheaf J (Type w)) (X : Cᵒᵖ) :
((associatedFunctor (yoneda ⋙ presheafToSheaf J (Type w))).obj F).obj X ≅
((sheafToPresheaf J (Type w)).obj F).obj X where
hom := ((sheafificationAdjunction J (Type w)).homEquiv (yoneda.obj X.unop) F).to
yonedaEquiv.toFun
inv := yonedaEquiv.invFun ≫
((sheafificationAdjunction J (Type w)).homEquiv (yoneda.obj X.unop) F).invFu
hom_inv_id :=
by tac constructs a term of the expected type by running the tactic(s) tac .
inv_hom_id := by
set f := ((sheafificationAdjunction J (Type w)).homEquiv (yoneda.obj X.unop) F
set g := ((sheafificationAdjunction J (Type w)).homEquiv (yoneda.obj X.unop) F
let hom := yoneda.obj X.unop → (sheafToPresheaf J (Type w)).obj F
rw [← Category.assoc]
Anthony Bordg, 2 months ago • add Grothendieck topo
apply Eq.trans (b := (yonedaEquiv.invFun ≫ f ≫ g) ≫ yonedaEquiv.toFun)
case h₁ => rw [← Category.assoc]
case h₂ => have eq : f ≫ g = 𝟙 hom := by
apply Equiv.self_comp_symm
rw [eq]
rw[Category.comp_id]
apply Equiv.self_comp_symm
noncomputable def natTrans_associatedFunctor_sheafToPresheaf :
associatedFunctor (yoneda ⋙ presheafToSheaf J (Type w)) → sheafToPresheaf J (T
app F := { app := fun X => (associatedFunctor_iso_sheafToPresheaf_obj_obj J F X)
naturality := by
intros
apply funext
intro
unfold associatedFunctor_iso_sheafToPresheaf_obj_obj
simp only [Equiv.toFun_as_coe, types_comp_apply]
rw [yonedaEquiv_naturality']
simp only [EmbeddingLike.apply_eq_iff_eq]
apply Adjunction.homEquiv_naturality_left
Lean Infoview
GrothendieckTopos.lean:144:4
Tactic state
1 goal
C : Type w
inst✝ : SmallCategory C
J : GrothendieckTopology C
F : Sheaf J (Type w)
X : Cᵒᵖ
f : (yoneda.obj (Opposite.unop X) → (sheafToPresheaf J (Type w)).obj F) →
((presheafToSheaf J (Type w)).obj (yoneda.obj (Opposite.unop X)) → F) := ((sheafificationAdjunction J (Type w)).homEquiv (yoneda.obj (Opposite.unop X)) F).invFun
g : ((presheafToSheaf J (Type w)).obj (yoneda.obj (Opposite.unop X)) → F) →
(yoneda.obj (Opposite.unop X) → (sheafToPresheaf J (Type w)).obj F) := ((sheafificationAdjunction J (Type w)).homEquiv (yoneda.obj (Opposite.unop X)) F).toFun
hom : Type w := yoneda.obj (Opposite.unop X) → (sheafToPresheaf J (Type w)).obj F
⊢ (yonedaEquiv.invFun ≫ f) ≫ g ≫ yonedaEquiv.toFun = 𝟙 (((sheafToPresheaf J (Type w)).obj F).obj X)
All Messages (0)
Restart File
subobject-classifier* Launchpad 0 0 0 -- INSERT -- Anthony Bordg, 2 months ago Ln 144, Col 5 Spaces: 2 UTF-8 LF lean4

Proof-by-Action

Solution: no-code interface for *proof assistants*

↪ more **graphical** and **gestural** paradigm

$$\underbrace{\text{Direct manipulation}}_{\text{Proofs}} \quad \text{of} \quad \underbrace{\text{Diagrams}}_{\text{Statements}}$$

Proof-by-Action

Solution: no-code interface for *proof assistants*

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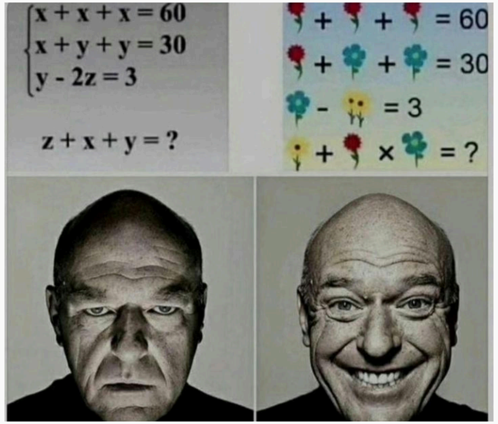
$$\underbrace{\text{Direct manipulation}}_{\text{Proofs}} \quad \text{of} \quad \underbrace{\text{Boxes}}_{\text{Statements}}$$

Proof-by-Action

Solution: no-code interface for *proof assistants*

↪ more **graphical** and **gestural** paradigm

Direct manipulation (Proofs) of Flowers 🌺 (Statements)



Origins: Existential Graphs

Existential Graphs (Peirce, 1896)

Three diagrammatic proof systems for **classical** logic:

- `Alpha`: *propositional* logic
- `Beta`: *first-order* logic
- `Gamma`: *higher-order* and *modal* logics

Existential Graphs (Peirce, 1896)

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The three icons of `Alpha`

- Sheet of assertion

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- Sheet of assertion

$$a$$

The three icons of `Alpha`

- Sheet of assertion

$$a \qquad \mapsto \qquad a \text{ is true}$$

The three icons of `Alpha`

- Sheet of assertion

$$
\begin{array}{ccc}
a & \mapsto & a \text{ is true} \\
 & \mapsto & \top \text{ (no assertion)}
\end{array}
$$

The three icons of `Alpha`

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$$a \quad \mapsto \quad a \text{ is true}$$

$$\quad \mapsto \quad \top \text{ (no assertion)}$$

- Juxtaposition

$$G \quad H$$

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$$G \quad H \quad \mapsto \quad G \text{ and } H \text{ are true}$$

The three icons of Alpha

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$$G \quad H \quad \mapsto \quad G \text{ and } H \text{ are true}$$

- Cut

The three icons of Alpha

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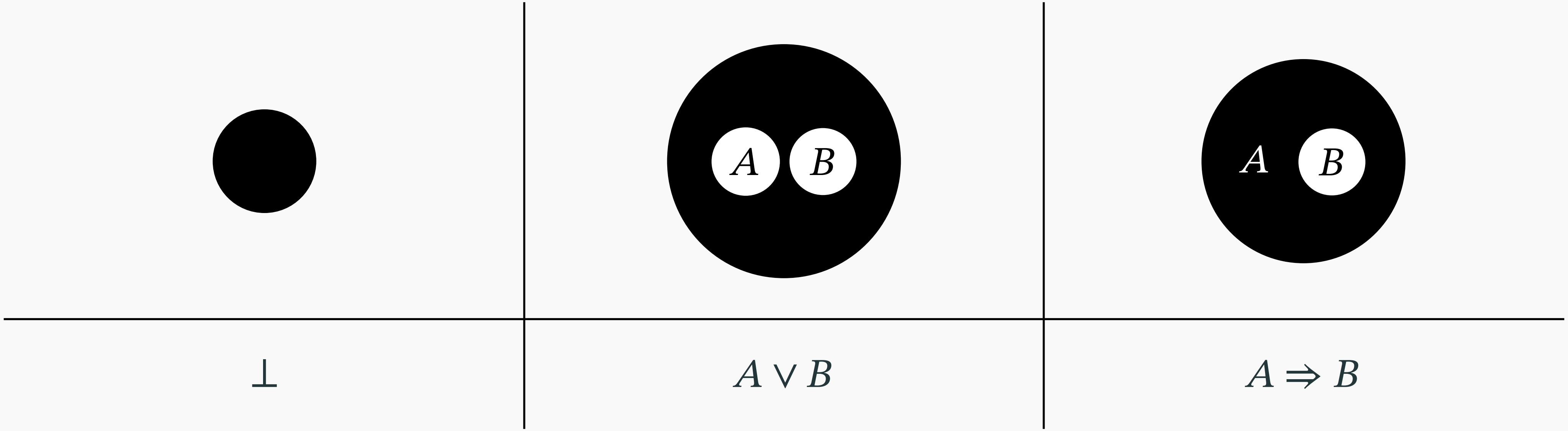
- Juxtaposition

$$G \quad H \quad \mapsto \quad G \text{ and } H \text{ are true}$$

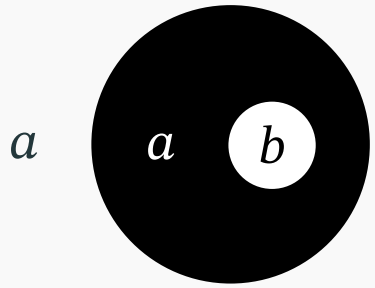
- Cut

$$\mapsto \quad G \text{ is not true}$$

Relationship with formulas



Example: Illative transformations



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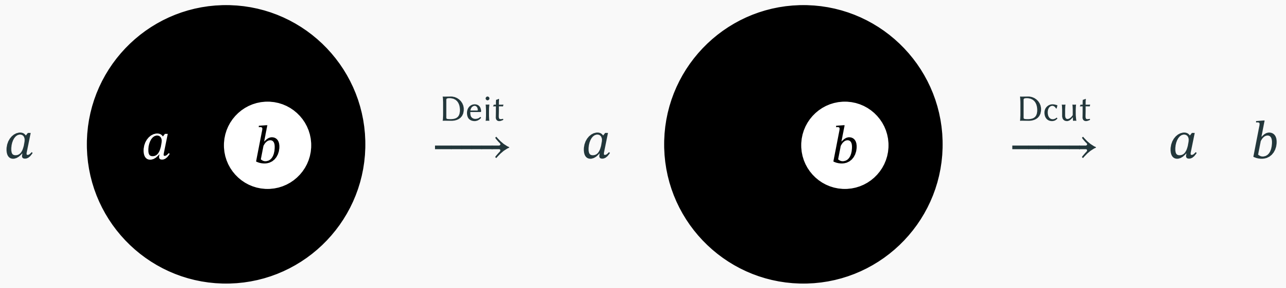

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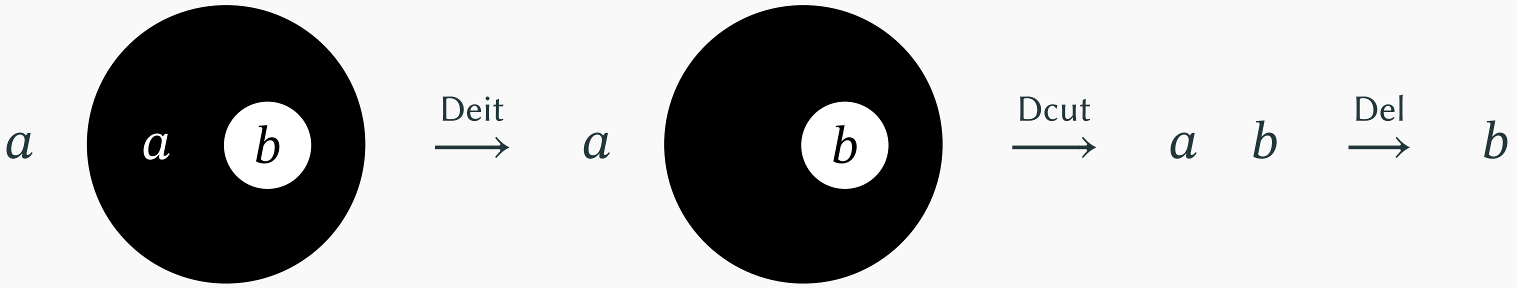
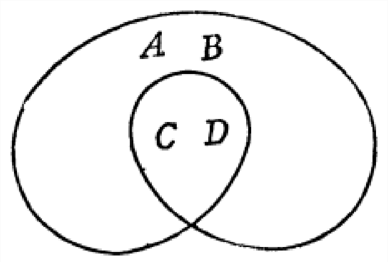


Intuitionistic Logic: Flowers

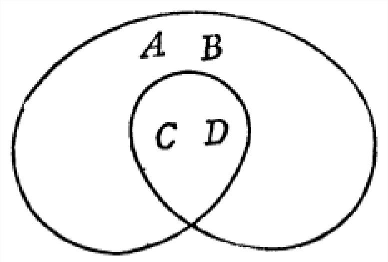
The scroll



I thought I ought to take the general form of argument as the basal form of composition of signs in my diagrammatization; and this necessarily took the form of a “scroll”, that is [...] a curved line without contrary flexure and returning into itself after once crossing itself.

— (Peirce 1906, pp. 533-534)

The scroll



$A \wedge B \Rightarrow C \wedge D$

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- "conditional de inesse" = **classical** implication

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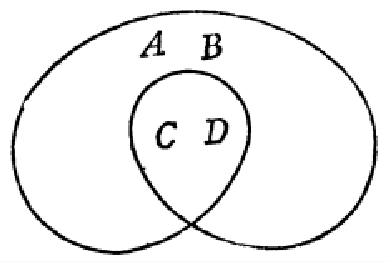

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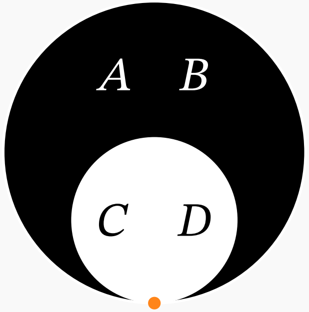

$\neg(A \wedge B \wedge \neg(C \wedge D))$

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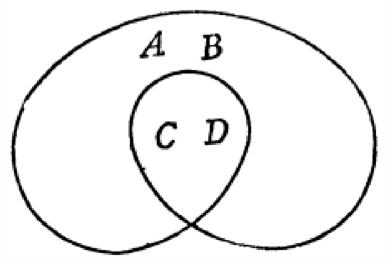
↪ scroll = two *nested cuts*

The scroll



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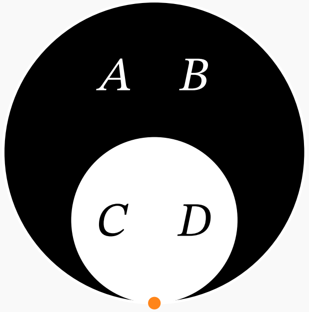

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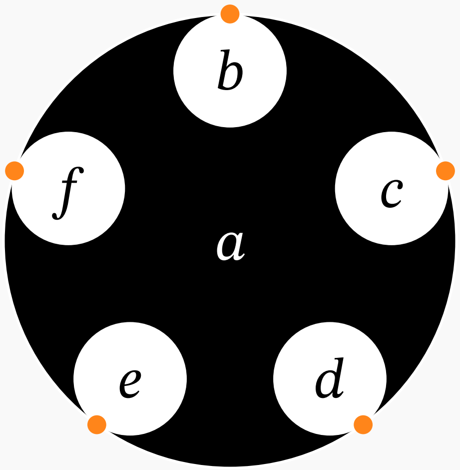
- "conditional de inesse" = **classical** implication

↪ scroll = two *nested cuts*

- Peirce also introduced $\Rightarrow$ in logic! (Lewis 1920, p. 79)

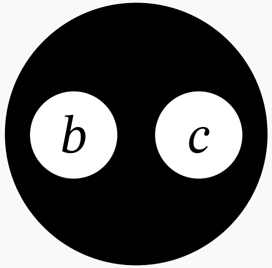
The n-ary scroll (Oostra 2010)



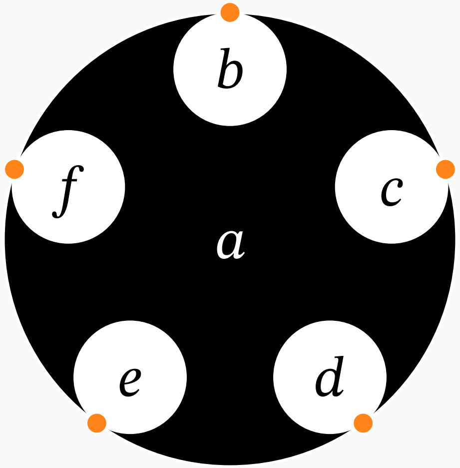
$n = 5$

The n-ary scroll (Oostra 2010)

Classical



$b \vee c$



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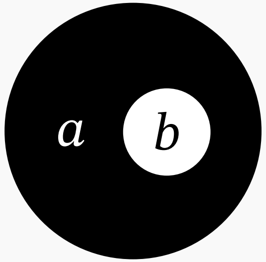
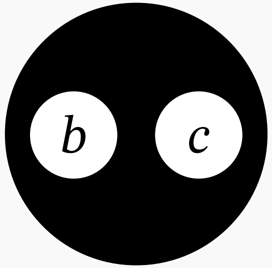


$a \Rightarrow b$

The n-ary scroll (Oostra 2010)

Classical



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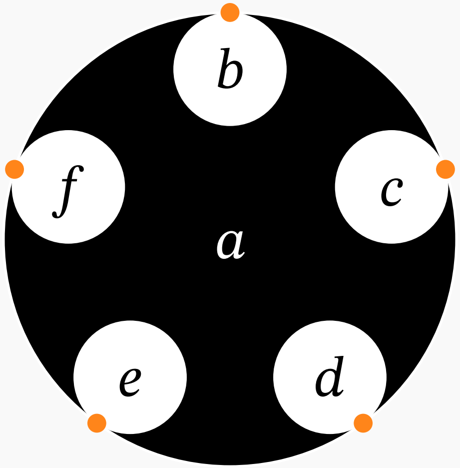
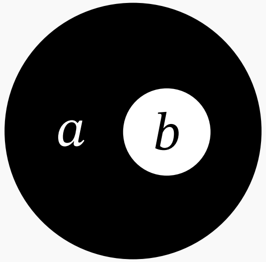


$a \Rightarrow b \vee c \vee d \vee e \vee f$

$n = 5$

Classical

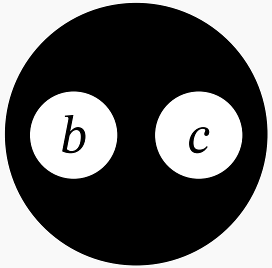


$a \Rightarrow b$

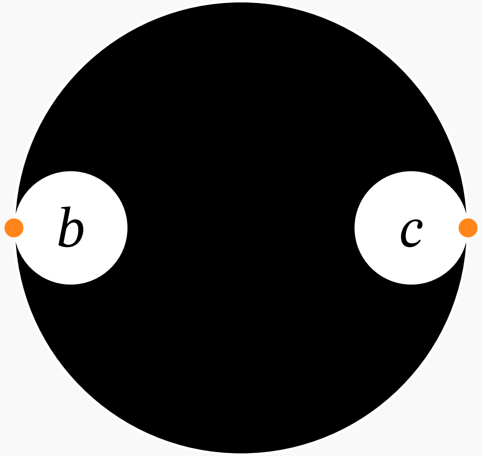
The n-ary scroll (Oostra 2010)

Continuity!

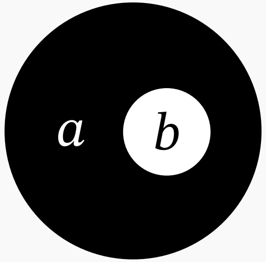
Intuitionistic



$\neq$



Classical



$\neg(\neg b \wedge \neg c)$

$b \vee c$

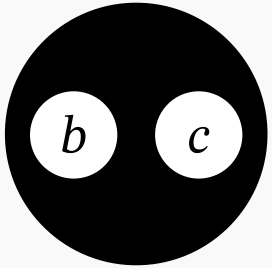
$n = 2$

$a \Rightarrow b$

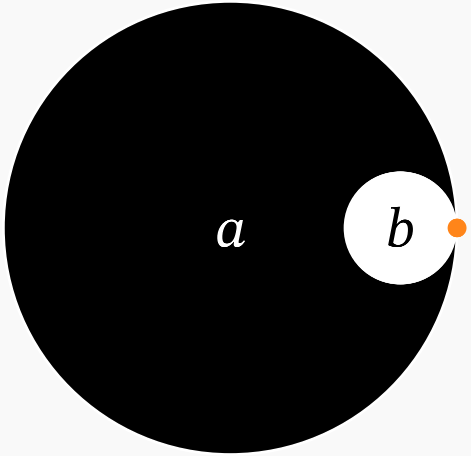
The n-ary scroll (Oostra 2010)

Continuity! *Generalizes* Peirce's **scroll**

Intuitionistic



$\neg(\neg b \land \neg c)$

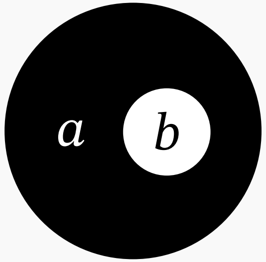


$a \Rightarrow b$

$n = 1$

$\neq$

Intuitionistic

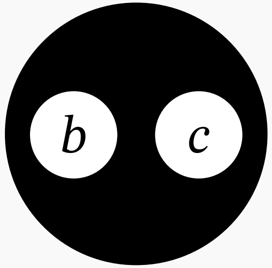


$\neg(a \land \neg b)$

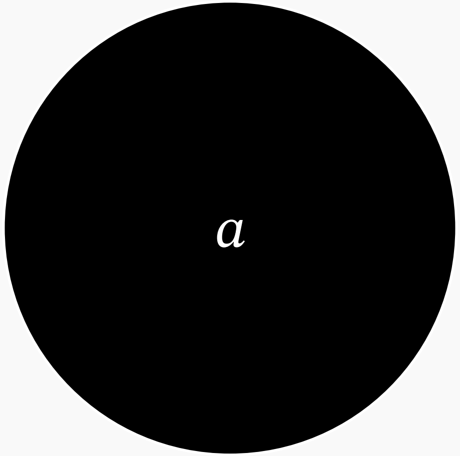
The n-ary scroll (Oostra 2010)

Continuity! *Generalizes* Peirce's **scroll** and **cut**

Intuitionistic



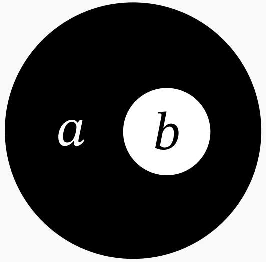
$\neg(\neg b \wedge \neg c)$



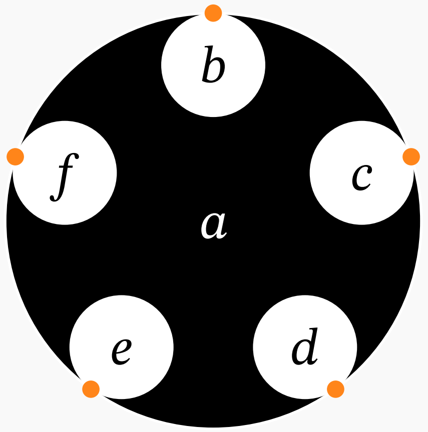
$\neg a \triangleq a \Rightarrow \bot$

$n = 0$

Intuitionistic

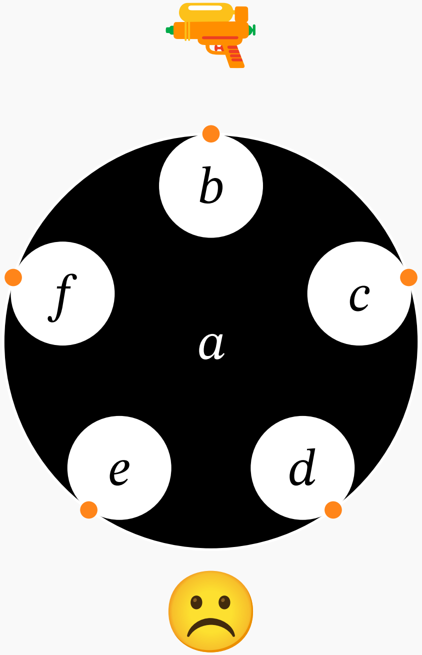


$\neg(a \wedge \neg b)$


b
f
c
a
e
d

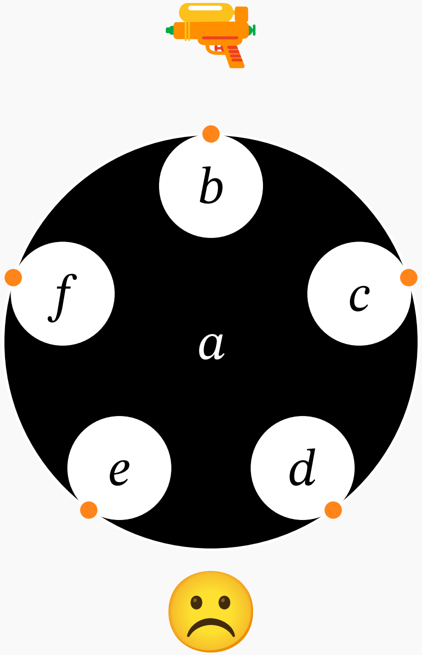
Blooming

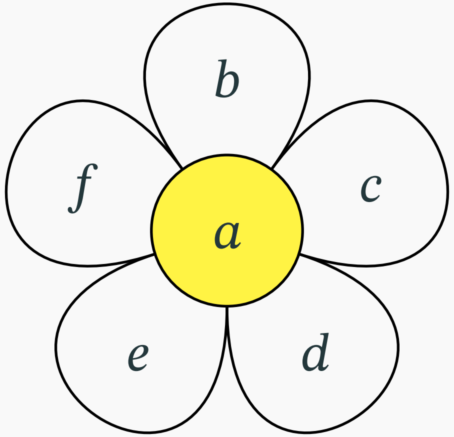
(Me, 2022)



Blooming

(Me, 2022)

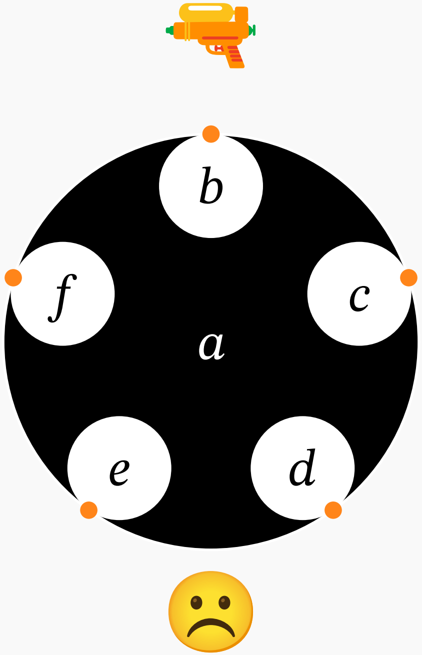


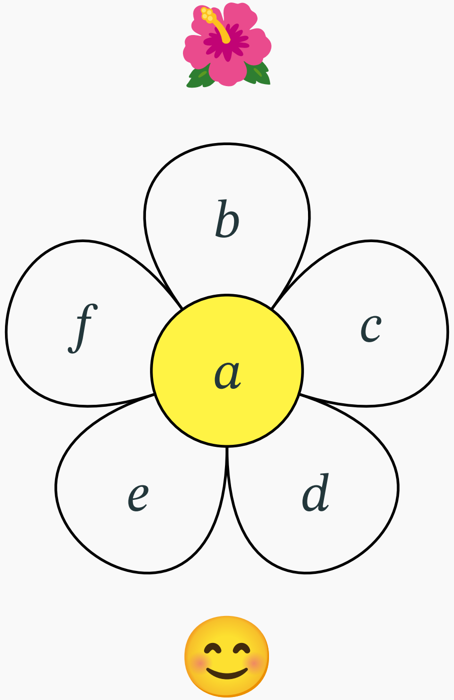


Turn **inloops** into petals.

Blooming

(Me, 2022)





<u>"Make love, not war"</u>

Corollaries

The original "theorems" of geometry were those propositions that Euclid proved, while the **corollaries** were simple deductions from the theorems inserted by Euclid's commentators and editors. They are said to have been marked the figure of a little garland (or corolla), in the origin.

— Peirce, MS 514 (1909) (Peirce 1976)

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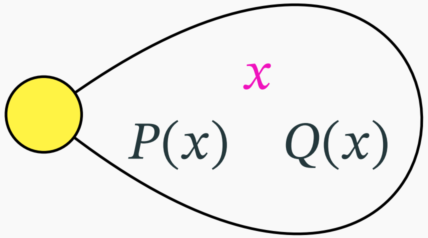
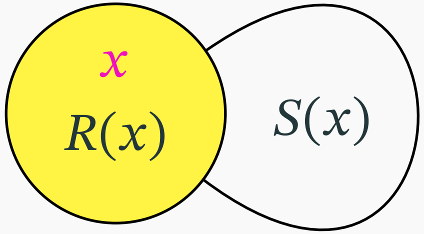
Petals = (possible) **corolla**-ries of pistil!

Gardens

$\exists / \forall$ = binder in petal/pistil





$\exists x.P(x) \wedge Q(x)$

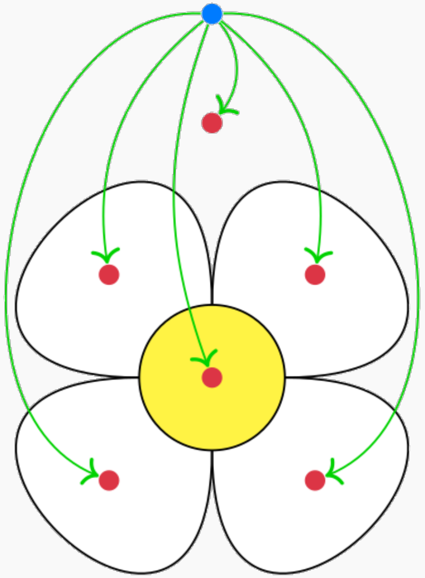
$\forall x.R(x) \Rightarrow S(x)$

garden = content of an area (binders + flowers)

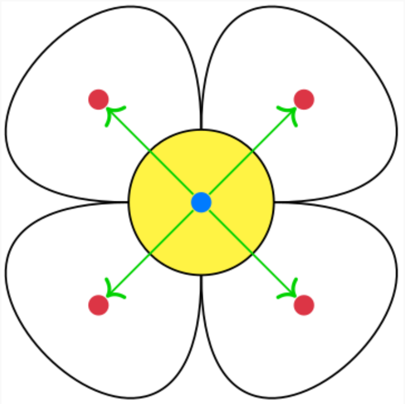
Reasoning with Flowers

Iteration and Deiteration

Justify a target flower by a source flower



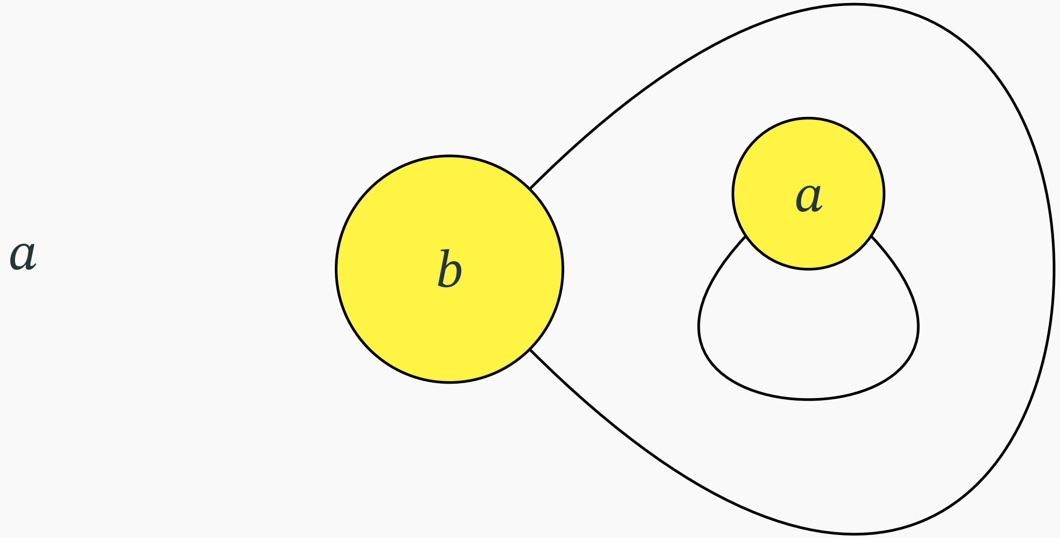
cross-pollination



self-pollination

Iteration and Deiteration

Works at **arbitrary** depth!



Cross-pollination

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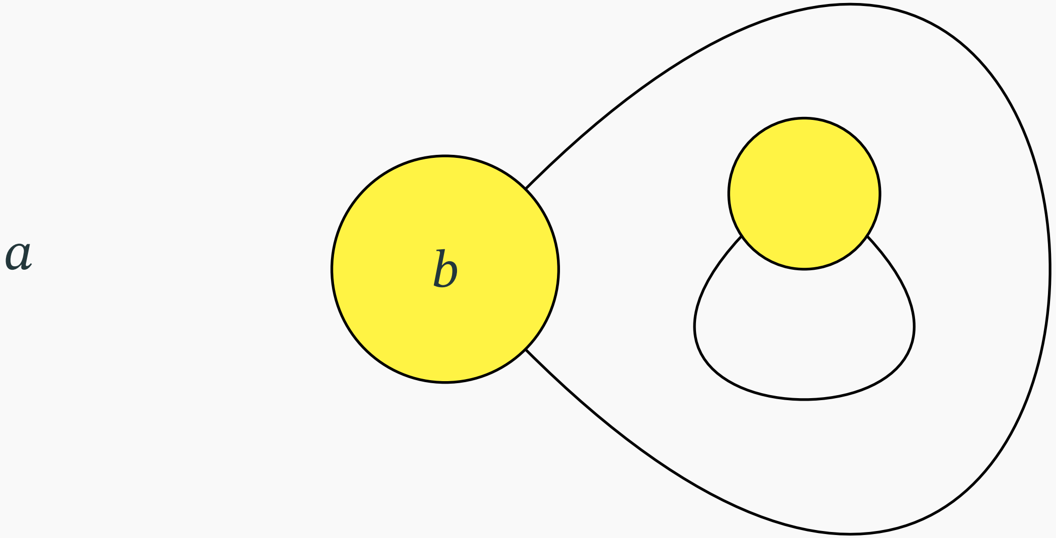

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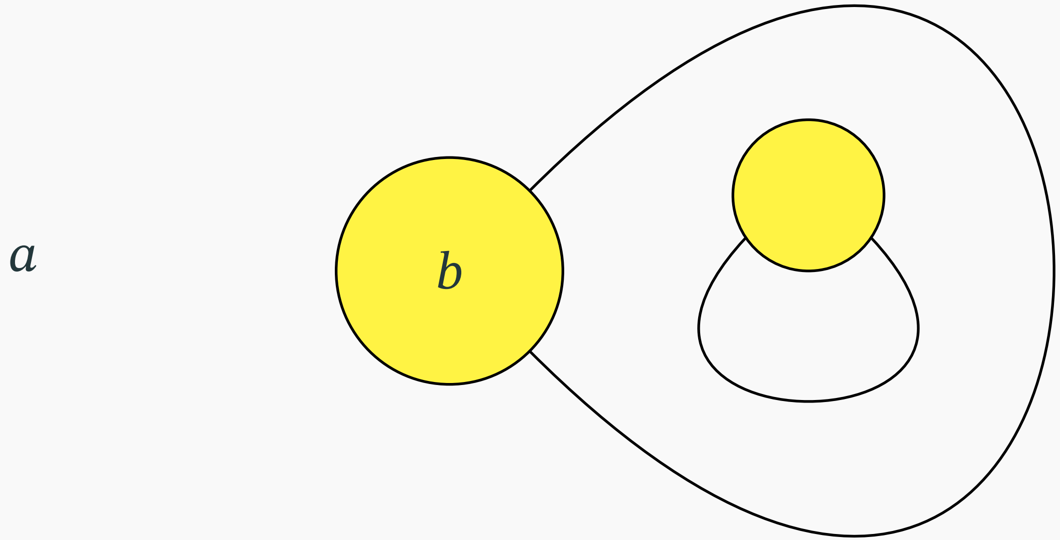

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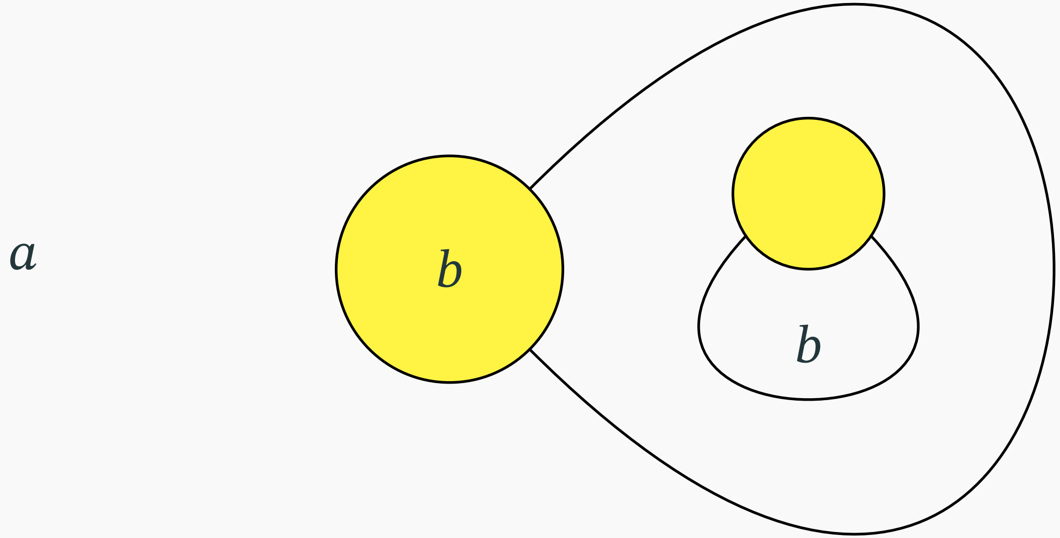
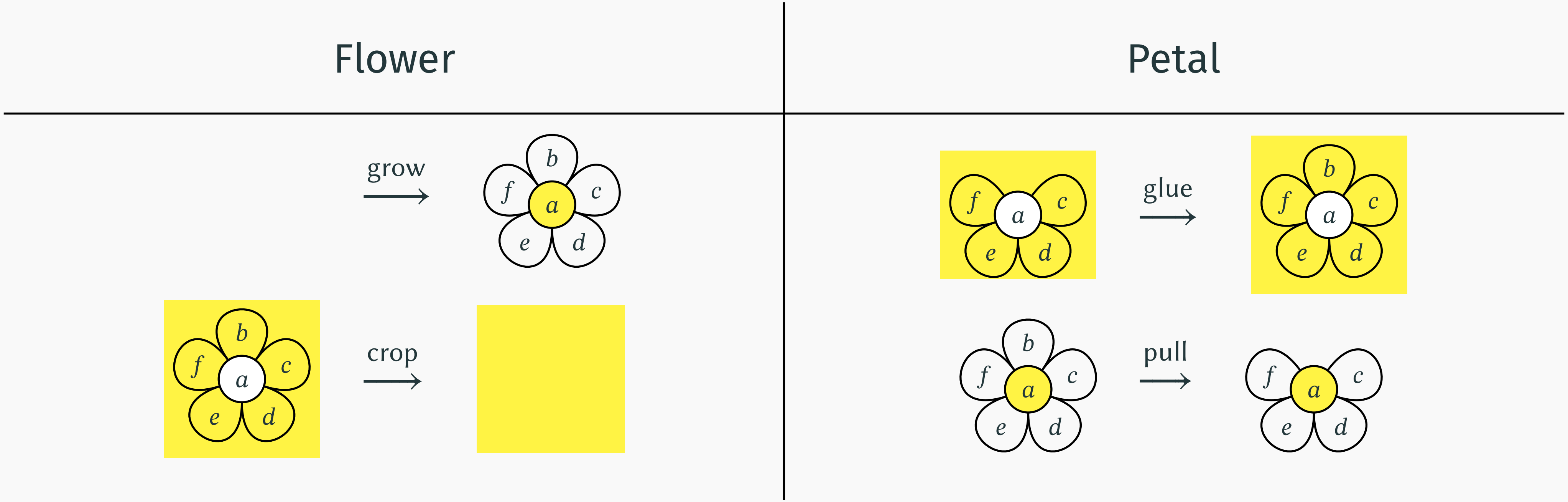


Self-pollination

Insertion and Deletion

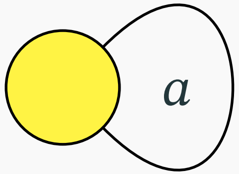
Split in two:



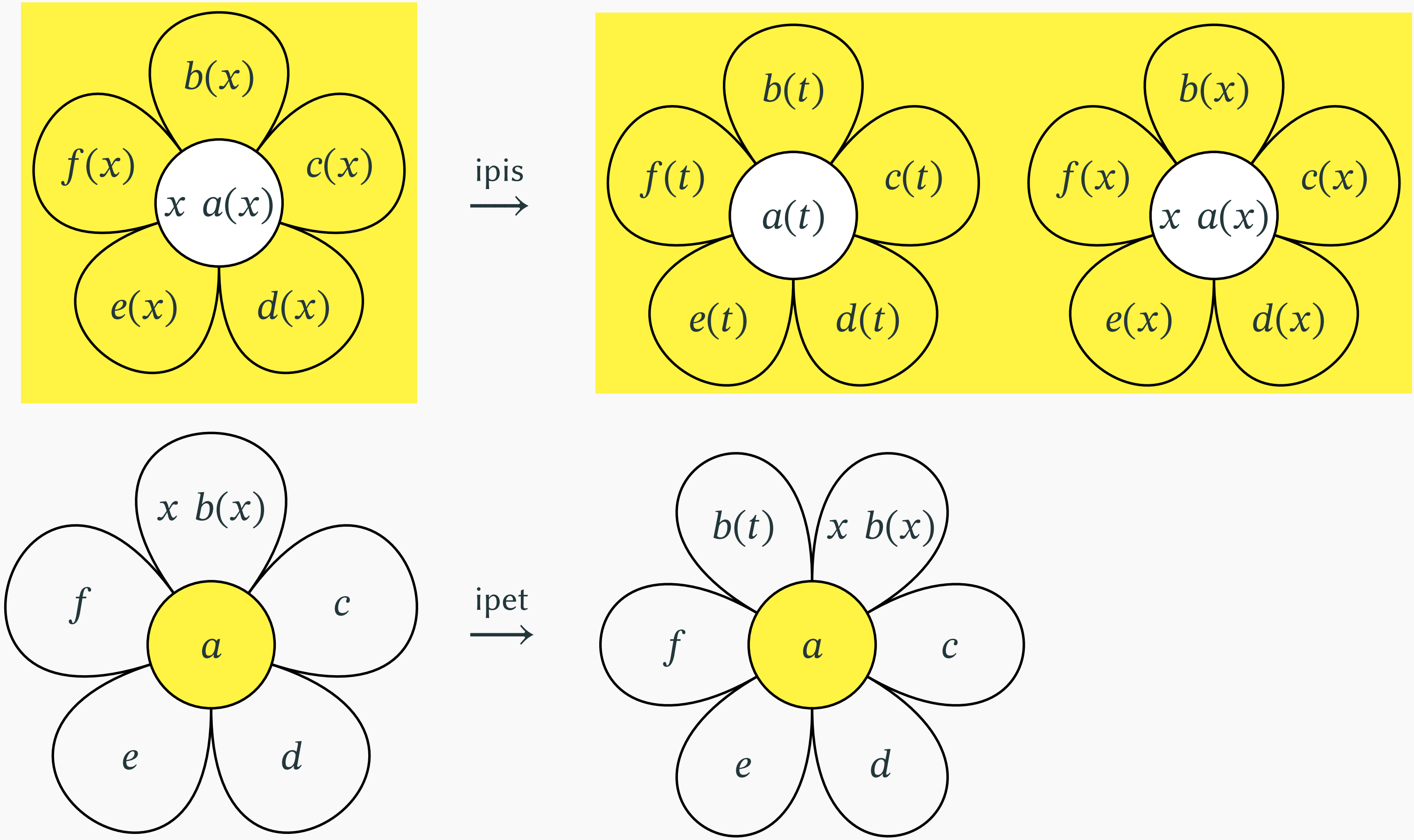
Backward reading: conclusion ⟶ premiss

Scrolling

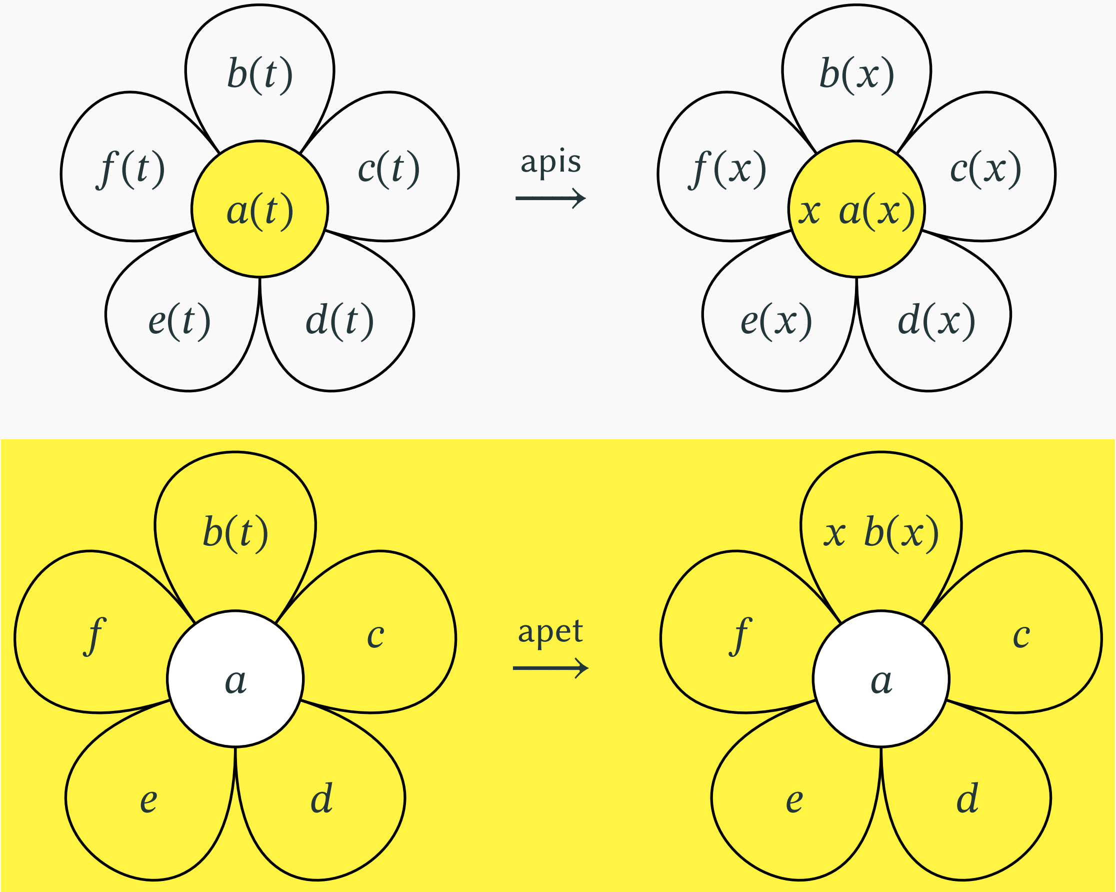
Intuitionistic restriction of **double-cut** principle:

$a \xrightarrow{\text{epis}}$ 

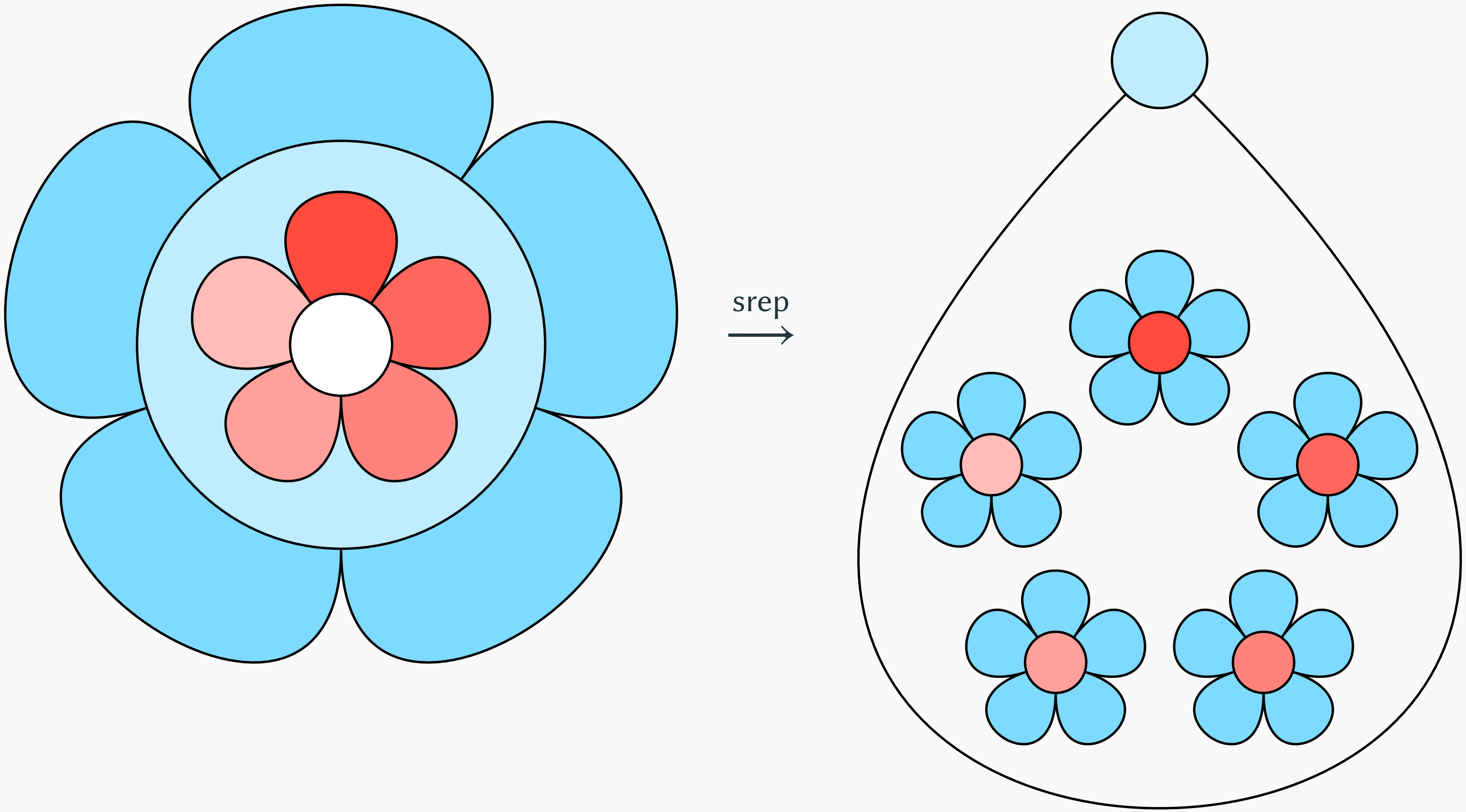
Instantiation



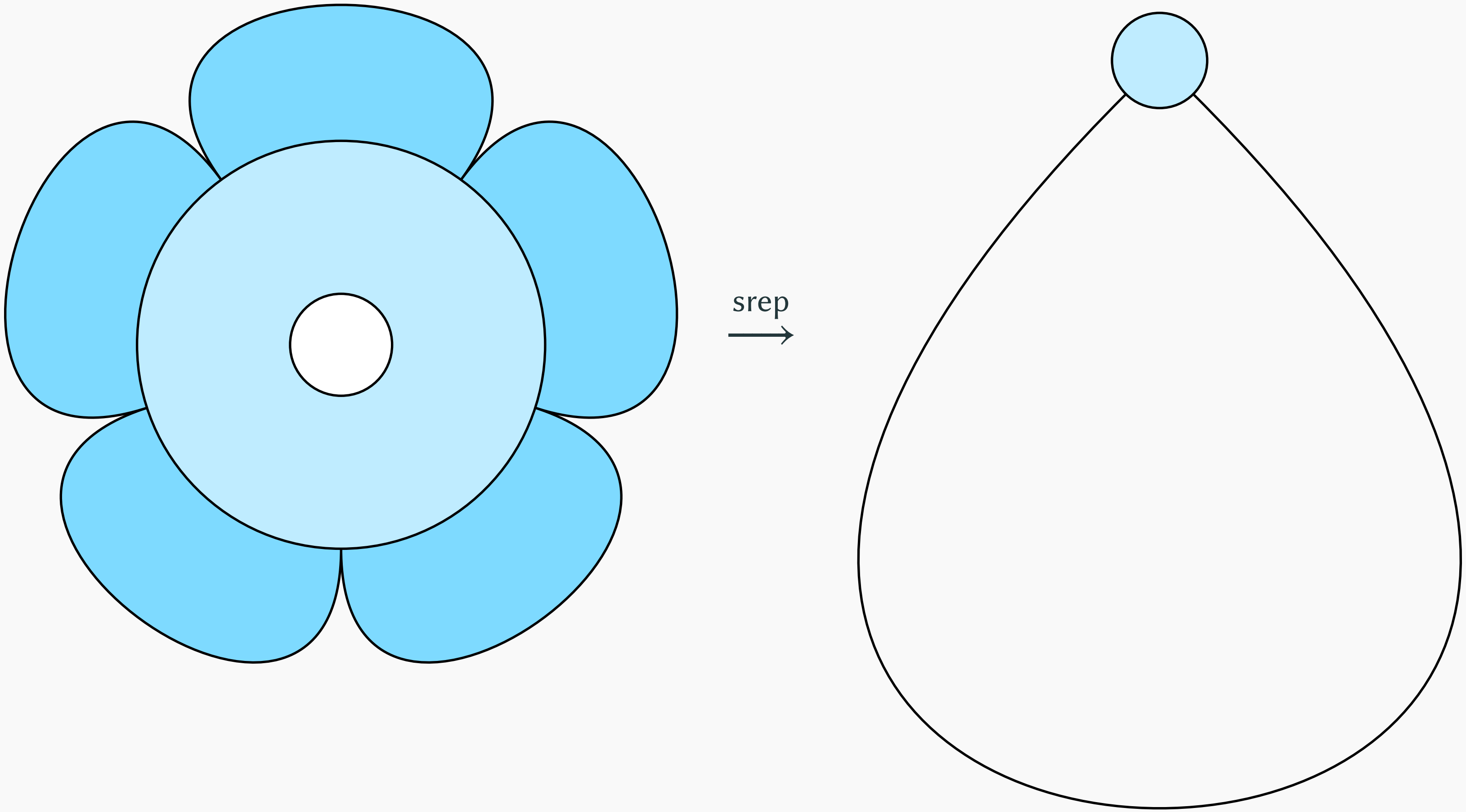
Abstraction



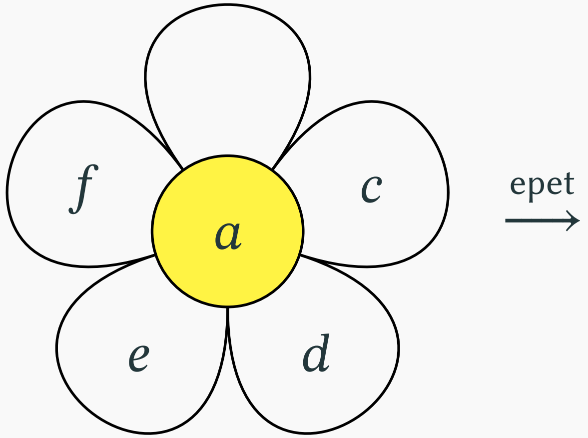
Case reasoning



Ex falso quodlibet



QED



Metatheory: Nature vs. Culture

Natural rules ❀

$$❀ = \underbrace{\text{(De)iteration}}_{\{\text{poll}\downarrow, \text{poll}\uparrow\}} \cup \underbrace{\text{Instantiation}}_{\{\text{ipis}, \text{ipet}\}} \cup \underbrace{\text{Scrolling}}_{\{\text{epis}\}} \cup \underbrace{\text{QED}}_{\{\text{epet}\}} \cup \underbrace{\text{Case reasoning}}_{\{\text{srep}\}}$$

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Let Φ, Ψ be *bouquets*, i.e. multisets of flowers.

All rules are:

- **Invertible**: if $\Phi \longrightarrow \Psi$ then Ψ equivalent to Φ

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↪ Reduces proof-search space

Cultural rules ✂

$$✂ = \underbrace{\text{Insertion}}_{\{\text{grow},\text{glue}\}} \cup \underbrace{\text{Deletion}}_{\{\text{crop},\text{pull}\}} \cup \underbrace{\text{Abstraction}}_{\{\text{apis},\text{apet}\}}$$

Cultural rules ✂

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- All rules are **non-invertible**
- Some rules are **non-analytic**

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- Remember our paradigm:

$$\text{proving} = \text{erasing}$$

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- This works in arbitrary contexts X (i.e. one-holed bouquets)
- Formally:

Definition: For any bouquets Φ and Ψ, Ψ is *provable* from Φ, written $\Phi \vdash \Psi$, if for any context X in which Φ occurs and *pollinates* the hole of X, we have

$$X\boxed{\Psi} \longrightarrow X\boxed{}$$

Cult-elimination

Theorem (Soundness): If $\Phi \vdash \Psi$ then $\Phi \overset{\mathcal{K}}{\vDash} \Psi$ in every Kripke structure $\mathcal{K}$.

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Completeness of analytic fragment ❀!

The Flower Prover

*A **demo** is worth a thousand pictures!*

Flower Prover

- Represent flowers (logical statements) as nested **boxes**
- **Modal interface** to interpret gestural actions:

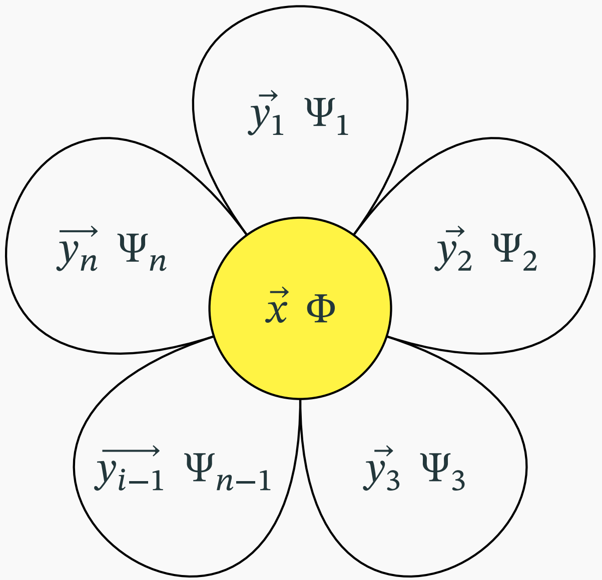
$$\text{Proof mode} \iff \text{Natural (invertible and analytic) rules}$$

$$\text{Edit mode} \iff \text{Cultural (non-invertible) rules}$$

$$\text{Navigation mode} \iff \text{Contextual closure (functoriality)}$$

Related works (non-exhaustive)

- **Structural proof theory:**
  - (Guenot 2013): rewriting-based **nested sequent** calculi
  - (Lyon 2021; Girlando et al. 2023): **fully invertible** labelled sequent calculi
- **Proof assistants:**
  - (Ayers 2021): Box datastructure similar to flowers
- **Categorical logic:**
  - (Johnstone 2002): coherent/geometric sequents in **topos theory**
  - (Bonchi et al. 2024): algebra of `Beta` (previous talk!)



$$\forall \vec{x}. \left(\bigwedge \Phi \Rightarrow \bigvee_i \exists \vec{y_i}. \Psi_i \right)$$

Future works

- Keep track of proof steps (*proof term*)

↪ extension to (higher-order, dependent) **type theory**

- Integration in **proof assistant** (e.g. Lean)

- Full-blown maths IDE for **mobile** devices (phones, tablets)

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Thank you!

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