Scroll nets

Towards a structural proof theory of Peirce's Existential Graphs

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Existential Graphs

Existential Graphs (EGs) (Peirce, 1896–1914)

Three *diagrammatic* proof systems for **classical** logic:

- **`Alpha`**: *propositional* logic
- **`Beta`**: *first-order* logic
- **`Gamma`**: *higher-order* and *modal* logics

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The three icons of `Alpha`

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$$a$$

- **Juxtaposition**

- **Separation** (sep)

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$$a \qquad \mapsto \qquad a \text{ is true}$$

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$$\begin{array}{ccc} a & \mapsto & a \text{ is true} \\ & \mapsto & \text{no assertion } (\textbf{trivially}\ \text{true}) \end{array}$$

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$$\Phi \quad \Psi$$

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$$\Phi \quad \Psi \quad \mapsto \quad \Phi \text{ is true } \textbf{and } \Psi \text{ is true}$$

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Φ

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\begin{array}{ccl}
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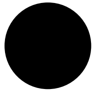
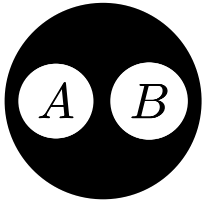
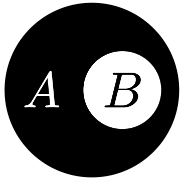
$$\Phi \quad \mapsto \quad \Phi \text{ is } \textbf{not} \text{ true}$$

Relationship with formulas

"Normal form" wrt. *associativity/commutativity* of $\{\wedge, \vee\}$ and *unitality* of $\wedge/\top$

	$A \quad B$	
$\top$	$A \wedge B$	$\neg A$

$\Rrightarrow$

		
$\bot$	$A \vee B$	$A \Rightarrow B$

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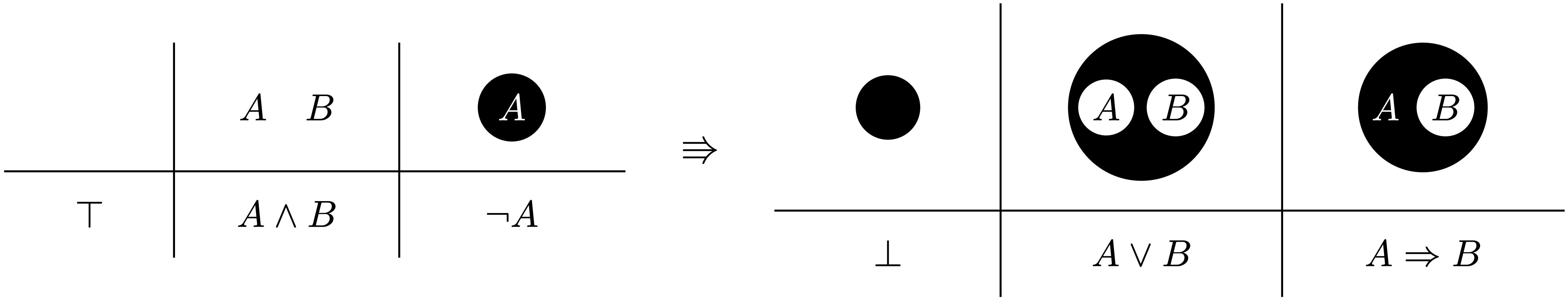

A "spatial" law, the **Double-sep** equation:

quotients further wrt. *unitality* of $\{\vee/\bot, \Rightarrow/\top\}$ and *currying* $(A \Rightarrow B \Rightarrow C \equiv A \wedge B \Rightarrow C)$

Illative transformations

Only 4 generalized **structural rules**, rewriting *premiss* $\rightarrow$ *conclusion*

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Iteration *co-contraction*			
$\Phi \quad \boxed{} \quad \to \quad \Phi \quad \boxed{\Phi}$			
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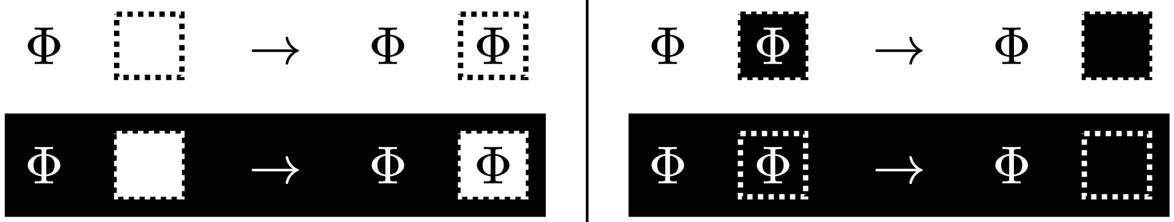
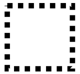
$\boxed{} \overset{\text{def}}{=}$ enclosed in arbitrary number of seps

Illative transformations

Only 4 generalized **structural rules**, rewriting *premiss* → *conclusion*

Iteration *co-contraction*	**Deiteration** *contraction*		



 $\stackrel{\text{def}}{=}$ enclosed in arbitrary number of seps

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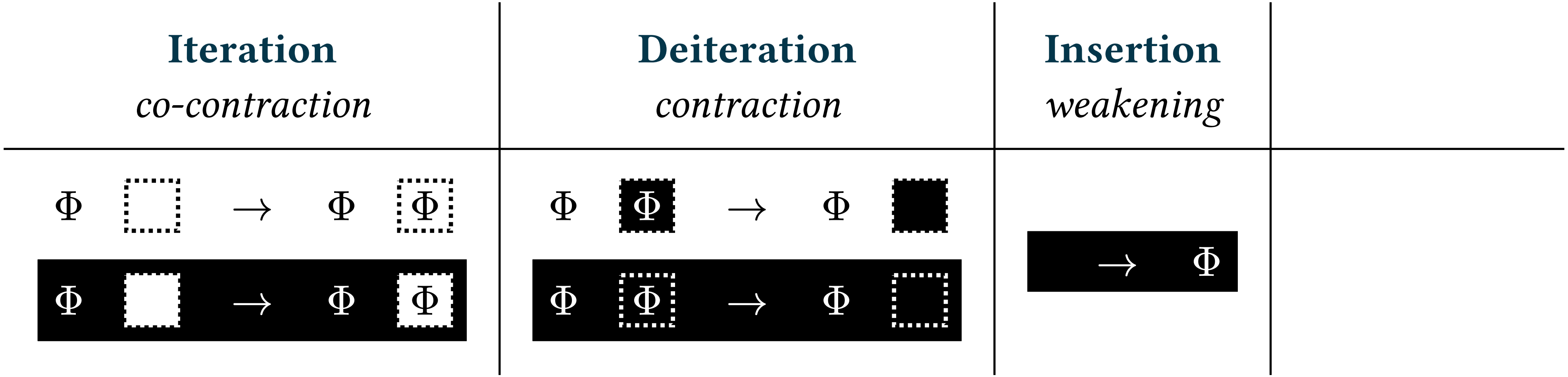
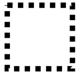
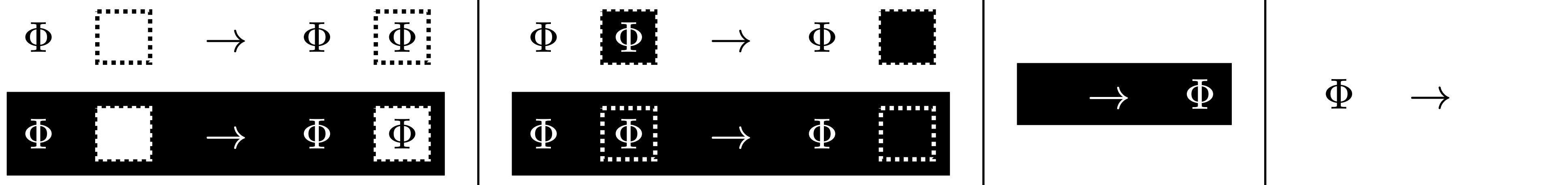


 $\overset{\text{def}}{=}$ enclosed in arbitrary number of seps

Illative transformations

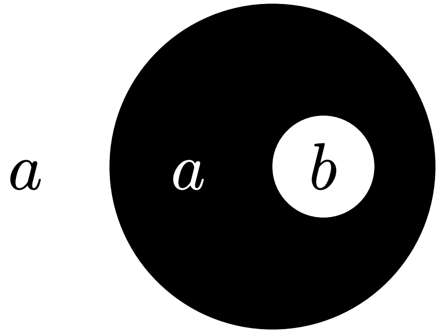
Only 4 generalized **structural rules**, rewriting *premiss* $\to$ *conclusion*

Iteration *co-contraction*	**Deiteration** *contraction*	**Insertion** *weakening*	**Deletion** *co-weakening*



$\boxed{} \overset{\text{def}}{=}$ enclosed in arbitrary number of seps

Example: modus ponens



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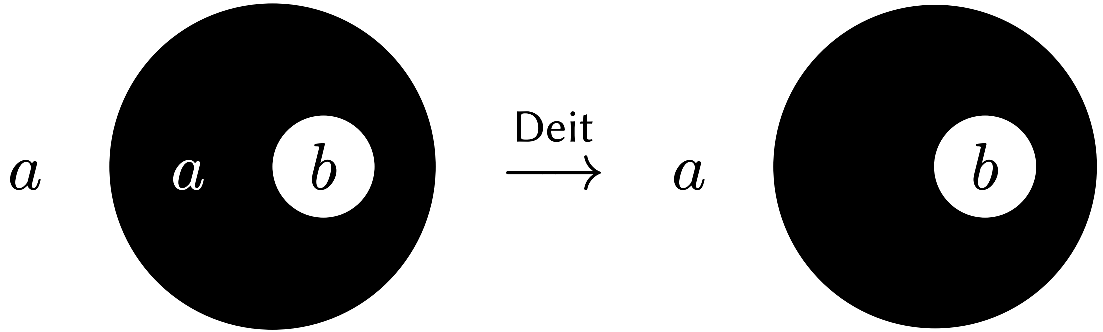

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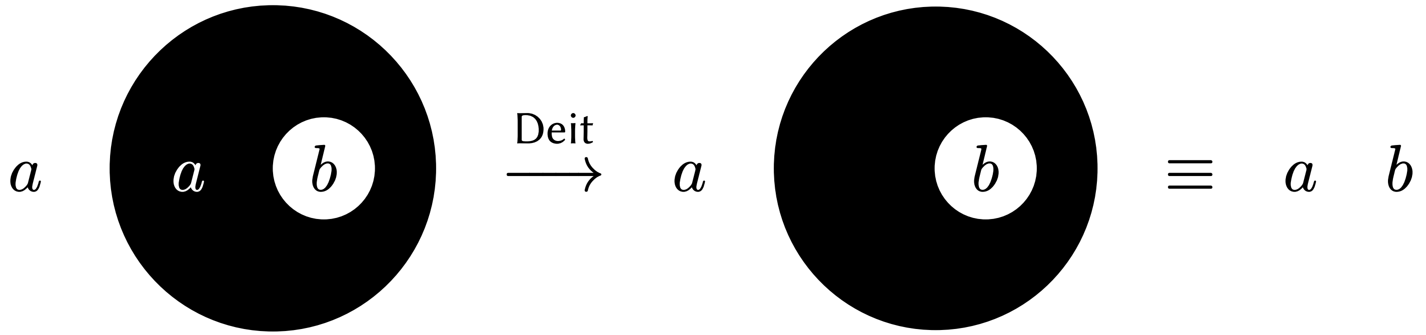

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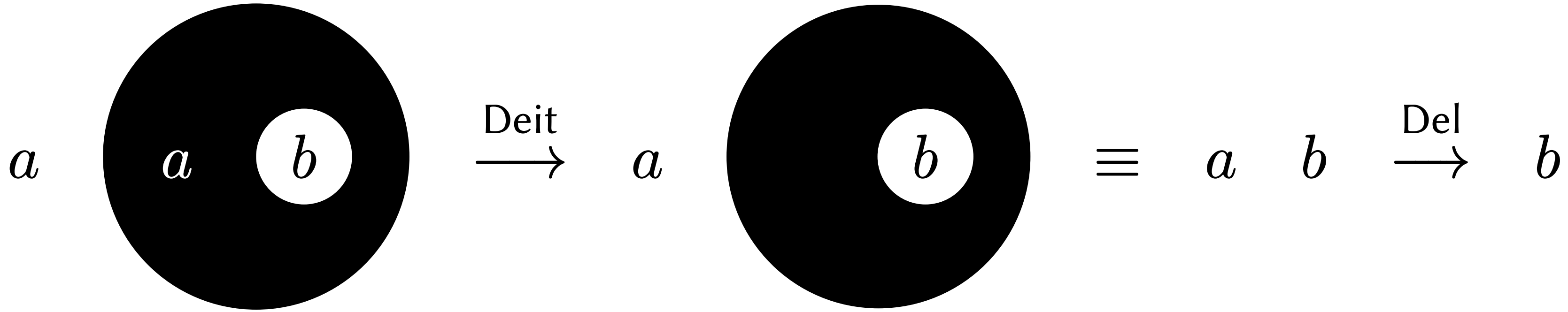
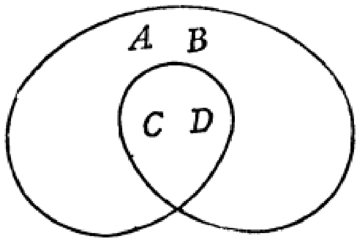


Scroll Nets

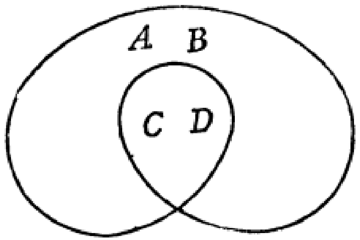
The scroll



I thought I ought to take the general form of argument as the basal form of composition of signs in my diagrammatization; and this necessarily took the form of a "scroll", that is [...] a curved line without contrary flexure and returning into itself after once crossing itself.

— (Peirce 1906, pp. 533-534)

The scroll



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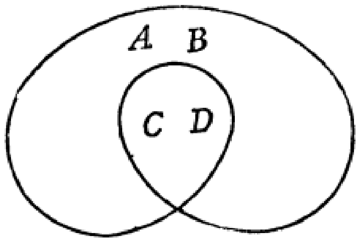

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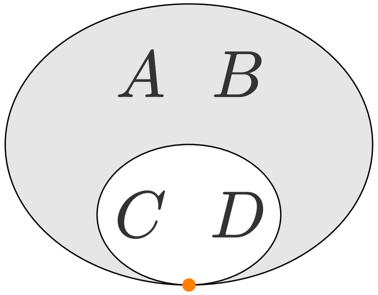

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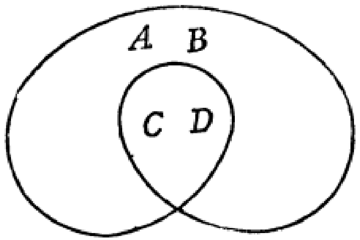

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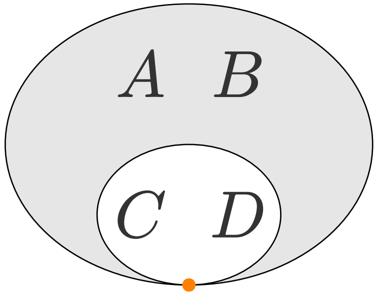

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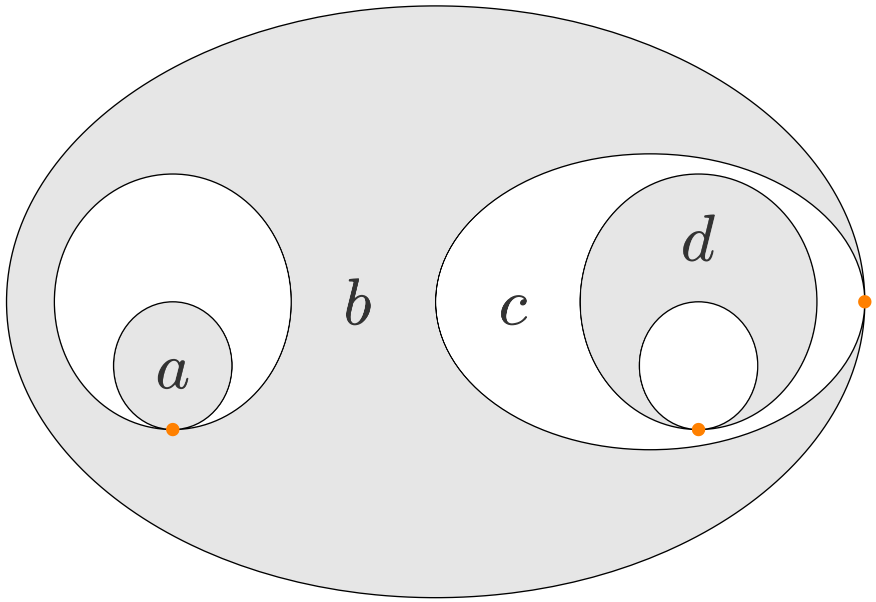
- Peirce also introduced '⇒' in logic! (Lewis 1920, p. 79)

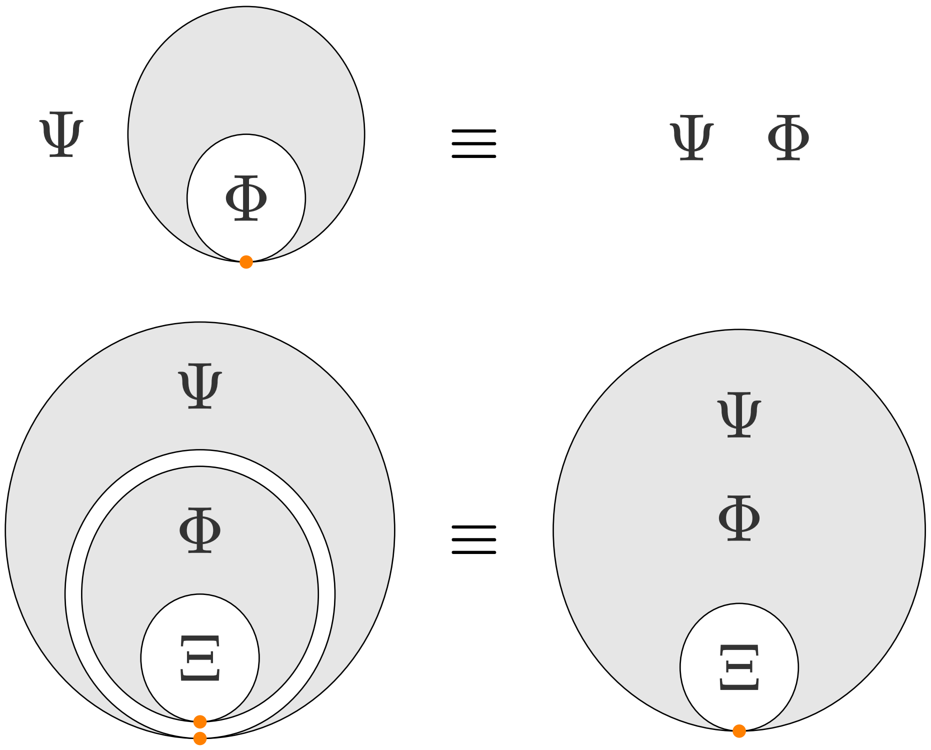
Scroll structures

Rest of this talk: only EGs built from **nested scrolls**, e.g.



This captures the $\{\top, \wedge, \Rightarrow\}$ fragment.

Double-sep law vs. Illative atomicity



[The double-sep law] cannot be assumed as an undeduced Permission, for the reason that it allows us to regard the Inner Scroll [Φ], after the Scroll is removed, as being a part of the Area [Ψ] on which the Scroll lies. Now this is not strictly either an Insertion or a Deletion; and **a perfectly analytical System of Permissions should permit only the indecomposable operations of Insertion and Deletion of Graphs** that are simple in expression.

— (Peirce 1906, p. 536)

The “more scientific way” — generalized (de)iteration

The more scientific way would be to substitute for [(de)iteration] and [the double-sep law] the following Permission:

If an Area, Υ, and an Area, Ω, be related in any of these four ways, viz., (1) Υ and Ω are the same Area; (2) If Ω is the Area of an Enclosure whose Place is Υ; (3) If Ω is the Area of an Enclosure whose Place is the Area of a second Enclosure whose Place is Υ; or (4) If Ω is the Place of an Enclosure whose Area is vacant except that it is the Place of an Enclosure whose Area is Υ; then, if Ω be a recto area, any simple Graph already scribed upon Υ may be iterated upon Ω; while if Ω be a verso Area, any simple Graph already scribed upon Υ and iterated upon Ω may be deiterated by being deleted or abolished from Ω.

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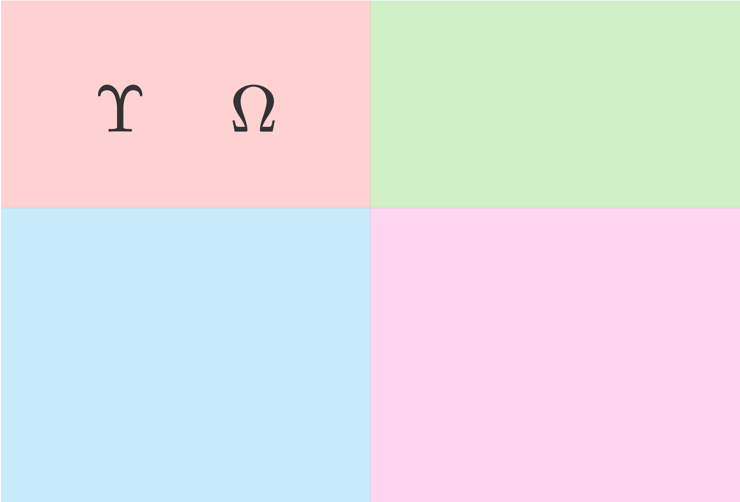

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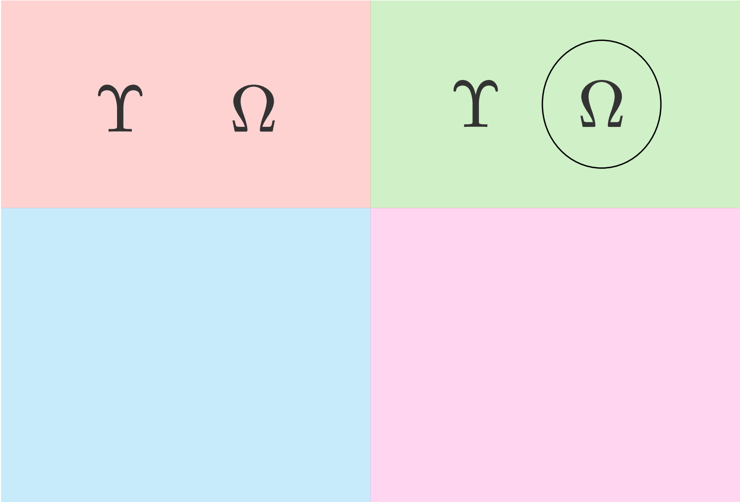

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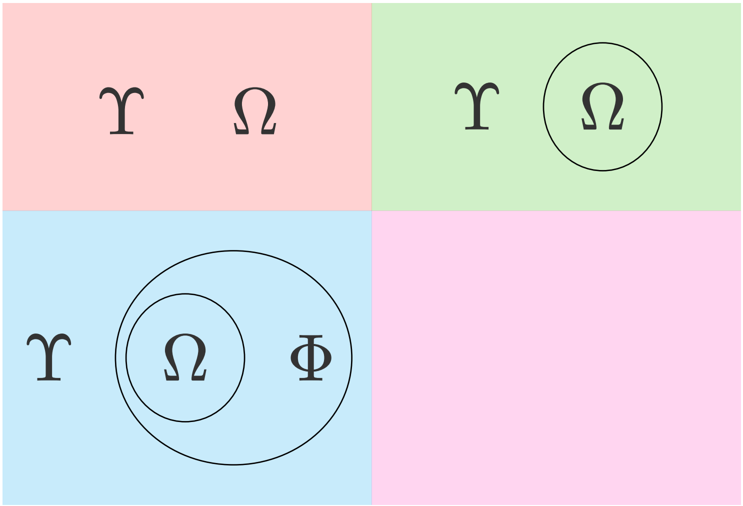

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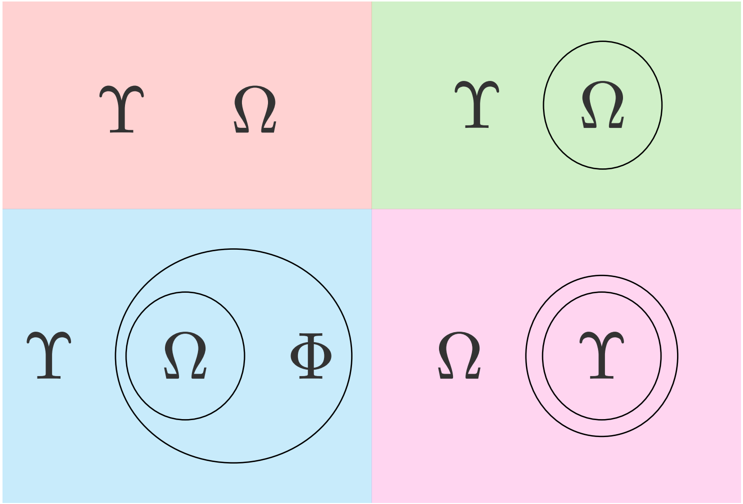

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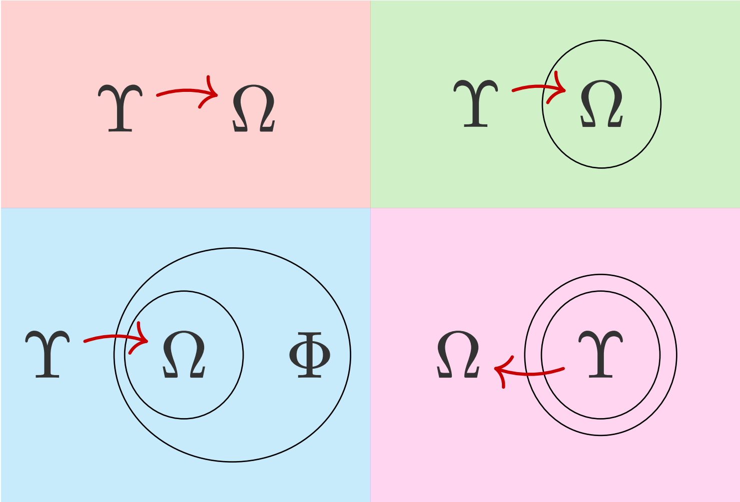

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A single "axiom"

> **These two Rules (of Deletion and Insertion, and of Iteration and Deiteration) are substantially all the undeduced Permissions needed**; the others being either Consequences or Explanations of these. Only, in order that this may be true, **it is necessary to assume that all indemonstrable implications of the Blank have from the beginning been scribed upon distant parts of the [Sheet of Assertion]**, upon any part of which they may, therefore, be iterated at will.
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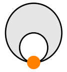
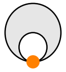
- Take  as the only "indemonstrable implication of the blank" (aka *axiom*)
- Derive the double-sep laws in both directions:



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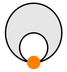
- Take  as the only “indemonstrable implication of the blank” (aka *axiom*)
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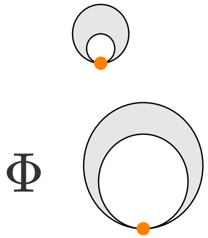

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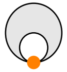
- Take  as the only “indemonstrable implication of the blank” (aka *axiom*)
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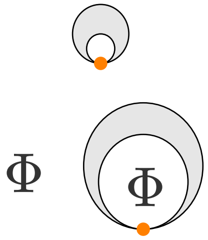

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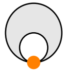
- Take  as the only “indemonstrable implication of the blank” (aka *axiom*)
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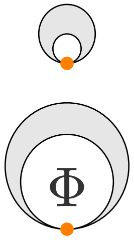

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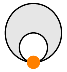
- Take  as the only “indemonstrable implication of the blank” (aka *axiom*)
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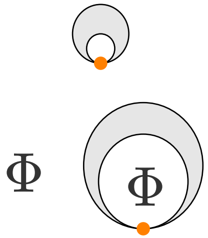
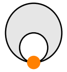


A single “axiom”

> **These two Rules (of Deletion and Insertion, and of Iteration and Deiteration) are substantially all the undeduced Permissions needed**; the others being either Consequences or Explanations of these. Only, in order that this may be true, **it is necessary to assume that all indemonstrable implications of the Blank have from the beginning been scribed upon distant parts of the [Sheet of Assertion]**, upon any part of which they may, therefore, be iterated at will.
>
> — (Peirce 1906, p. 537)

- Take  as the only “indemonstrable implication of the blank” (aka *axiom*)
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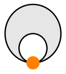
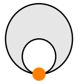


Φ

Recording illative transformations



Φ

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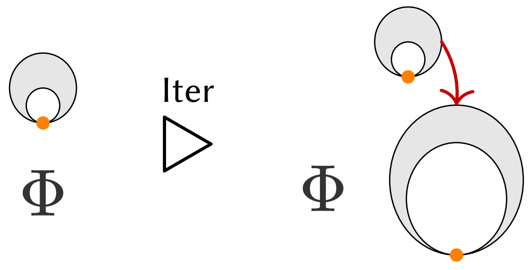

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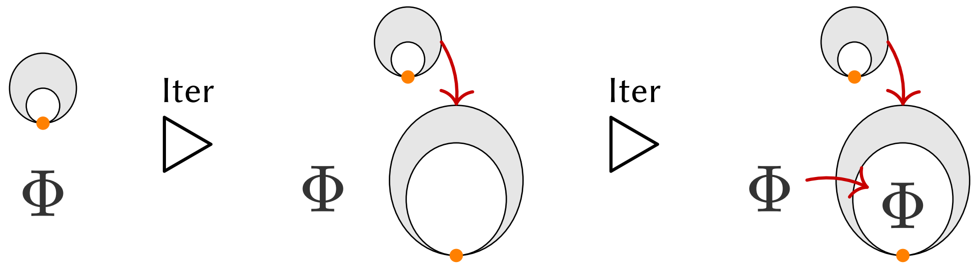

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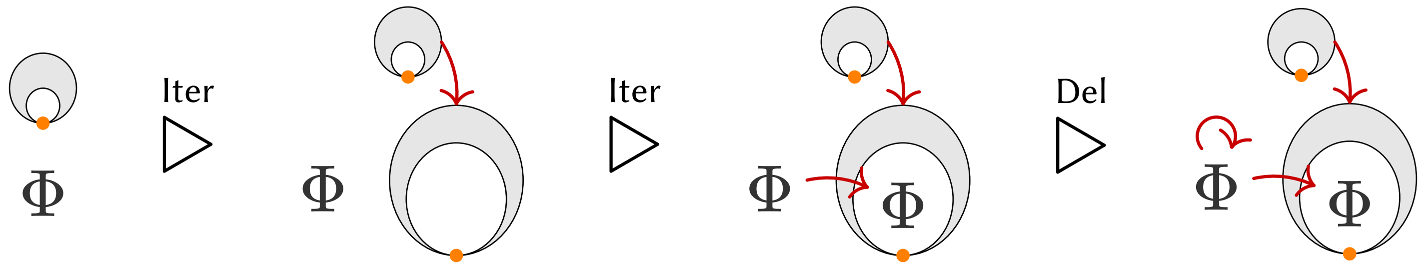

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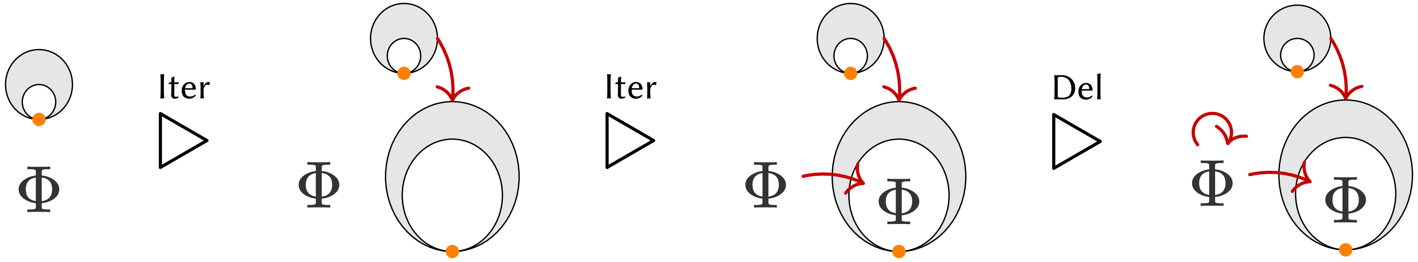
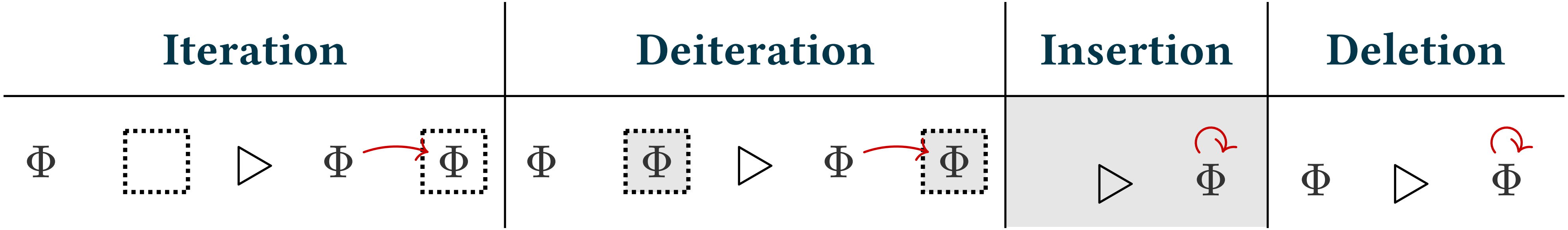
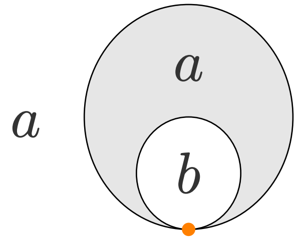




Term syntax

$$
\begin{aligned}
s, t, u &\Coloneqq a \mid [S(T)] && \text{(node)} \\
S, T, U &\Coloneqq J_1, \ldots, J_n && \text{(net)} \\
J &\Coloneqq x : s \mid \dot{x} : s \mid y/x : s \mid y/\dot{x} : s && \text{(judgment)} \\
J &\Coloneqq s \mid \cdot\, s \mid y/s \mid y/\cdot\, s && \text{(anonymous judgment)}
\end{aligned}
$$



$$a, [a(b)]$$

Term syntax

$$
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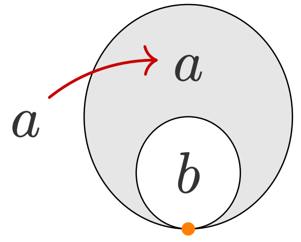

$$x : a, [x/a(b)]$$

Term syntax

$$
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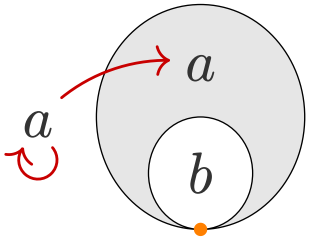

$$\dot{x} : a, [x/a(b)]$$

Term syntax

$$
\begin{aligned}
s, t, u &::= a \mid [S(T)] && \text{(node)} \\
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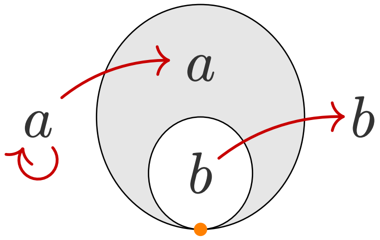

$$\dot{x} : a, [x/a(y : b)], y/b$$

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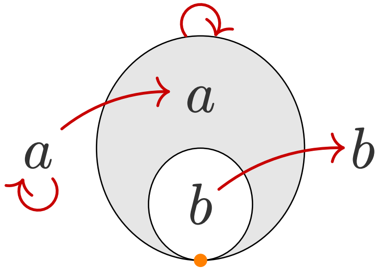

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Metatheory

Boundaries

Definition 1 (Boundaries) The *premiss* $\lceil S \rceil$ and *conclusion* $\lfloor S \rfloor$ of a scroll net S are defined mutually recursively by:

$$\lceil J_1, ..., J_n \rceil = \lceil J_1 \rceil, ..., \lceil J_n \rceil \qquad \lfloor J_1, ..., J_n \rfloor = \lfloor J_1 \rfloor, ..., \lfloor J_n \rfloor$$

$$\lceil a \rceil = a \qquad \lfloor a \rfloor = a$$

$$\lceil [S(T)] \rceil = [\lfloor S \rfloor (\lceil T \rceil)] \qquad \lfloor [S(T)] \rfloor = [\lceil S \rceil (\lfloor T \rfloor)]$$

$$\lceil x : s \rceil = x : \lceil s \rceil \qquad \lceil \dot{x} : s \rceil = \dot{x} : \lceil s \rceil \qquad \lfloor x : s \rfloor = x : \lfloor s \rfloor \qquad \lfloor y/x : s \rfloor = y/x : \lfloor s \rfloor$$

$$\lceil y/x : s \rceil = \lceil y/\dot{x} : s \rceil = \varnothing \qquad \lfloor \dot{x} : s \rfloor = \lfloor y/\dot{x} : s \rfloor = \varnothing$$

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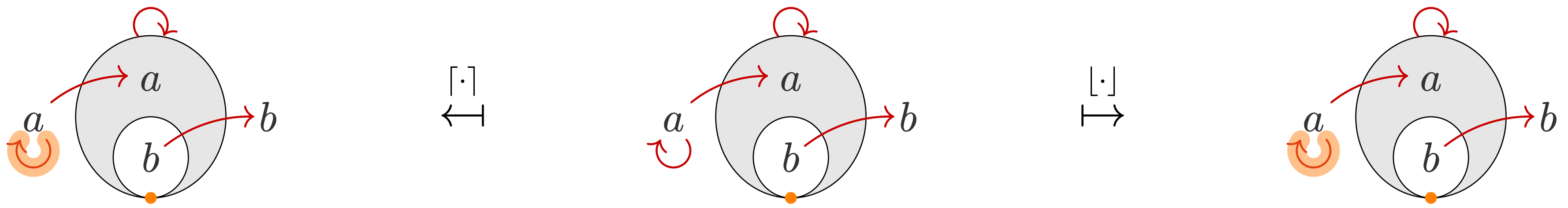

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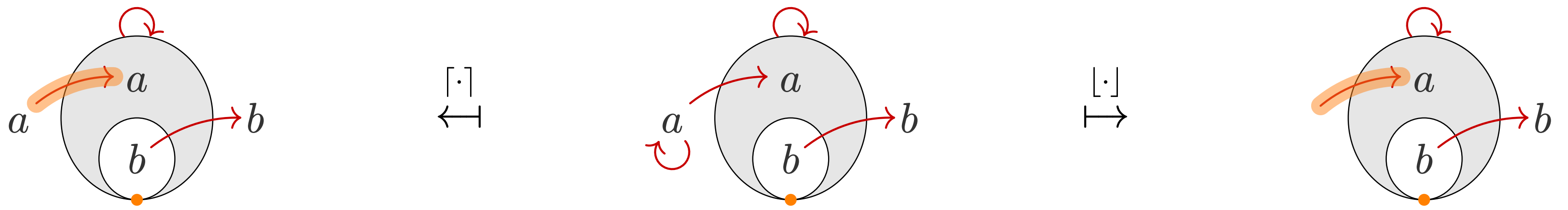

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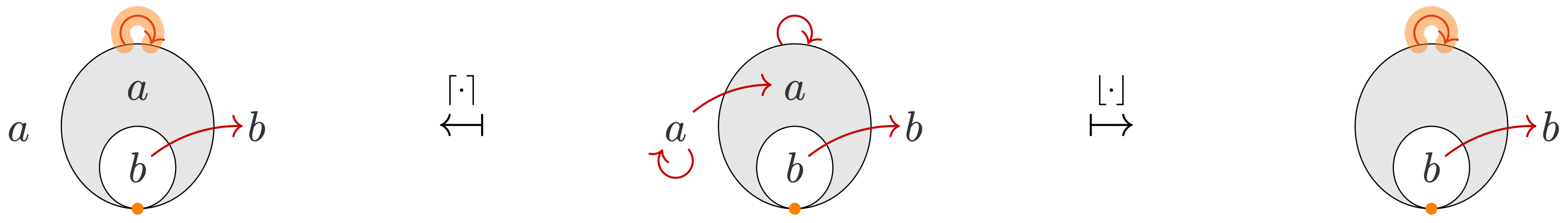
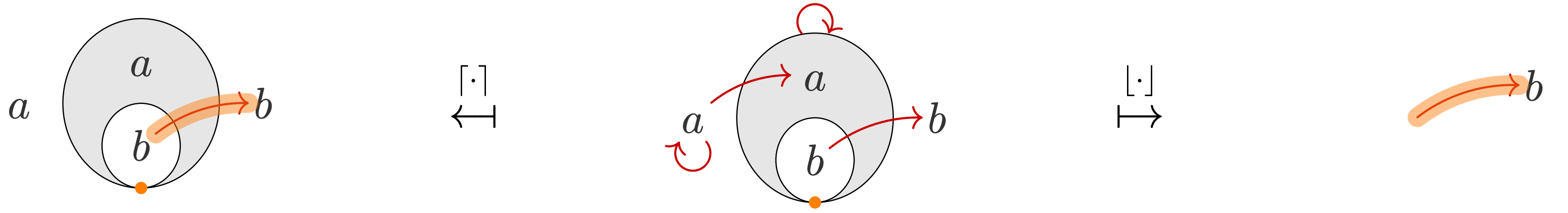


Boundaries

Definition 1 (Boundaries) The *premiss* $\lceil S \rceil$ and *conclusion* $\lfloor S \rfloor$ of a scroll net S are defined mutually recursively by:

$$
\begin{array}{cc}
\lceil J_1, ..., J_n \rceil = \lceil J_1 \rceil, ..., \lceil J_n \rceil & \lfloor J_1, ..., J_n \rfloor = \lfloor J_1 \rfloor, ..., \lfloor J_n \rfloor \\
\lceil a \rceil = a & \lfloor a \rfloor = a \\
\lceil [S(T)] \rceil = [\lfloor S \rfloor (\lceil T \rceil)] & \lfloor [S(T)] \rfloor = [\lceil S \rceil (\lfloor T \rfloor)] \\
\lceil x : s \rceil = x : \lceil s \rceil \qquad \lceil \dot{x} : s \rceil = \dot{x} : \lceil s \rceil & \lfloor x : s \rfloor = x : \lfloor s \rfloor \qquad \lfloor y/x : s \rfloor = y/x : \lfloor s \rfloor \\
\lceil y/x : s \rceil = \lceil y/\dot{x} : s \rceil = \varnothing & \lfloor \dot{x} : s \rfloor = \lfloor y/\dot{x} : s \rfloor = \varnothing
\end{array}
$$



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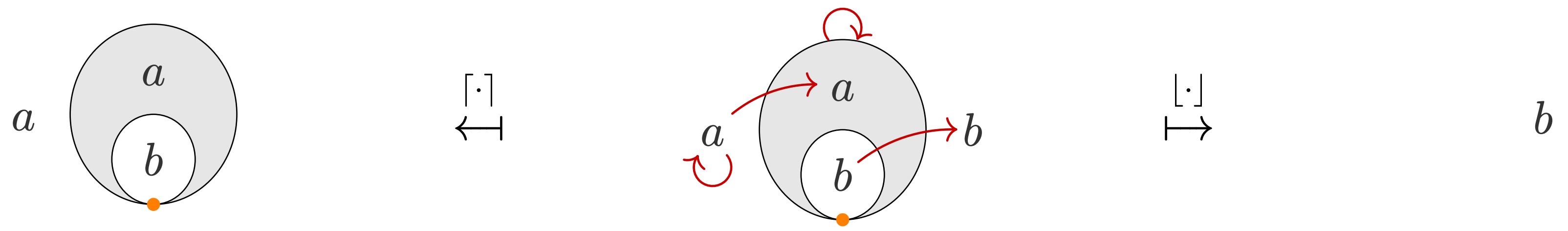

Soundness

Definition 2 (Correctness) A scroll net S is *correct* iff there exists a derivation $\lceil S \rceil \triangleright^* S$.

Theorem 3 If $S \triangleright T$ then $\lceil S \rceil = \lceil T \rceil$ and $[\![\lfloor S \rfloor]\!] \vDash [\![\lfloor T \rfloor]\!]$.

Corollary 3.1 (Soundness) If S is correct then $\lceil S \rceil \vDash \lfloor S \rfloor$.

Completeness

From every ND proof of $\Gamma \vdash A$, build a **correct** scroll net S with $\lceil S \rceil = \Gamma$ and $\lfloor S \rfloor = A$:

$\dfrac{}{\Gamma, x : A \vdash x : A}$		
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
$\dfrac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)N : B}$		

Completeness

From every ND proof of $\Gamma \vdash A$, build a **correct** scroll net S with $\lceil S \rceil = \Gamma$ and $\lfloor S \rfloor = A$:

$\dfrac{}{\Gamma, x : A \vdash x : A}$	$\Gamma \quad A$	$\Gamma \quad A$
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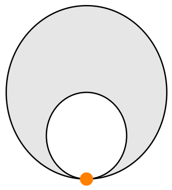
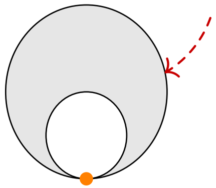
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$	Γ 	Γ 
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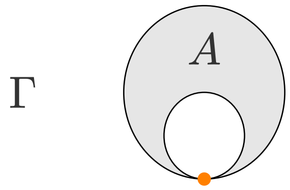
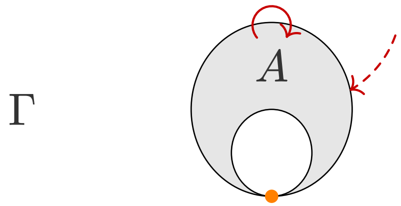
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
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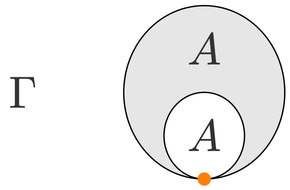
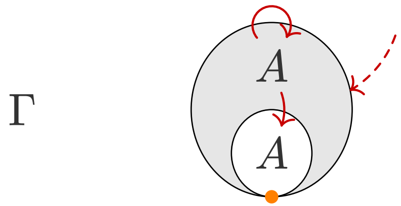
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
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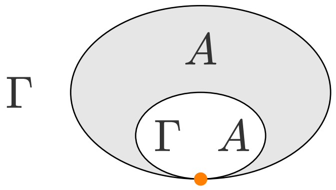
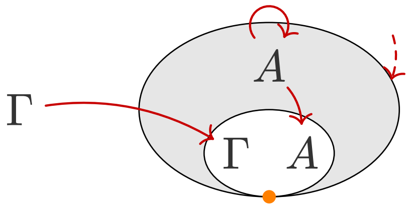
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
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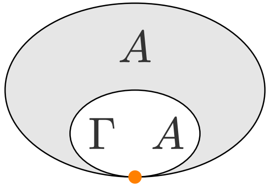
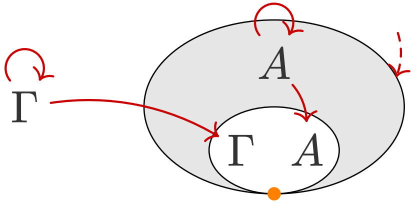
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
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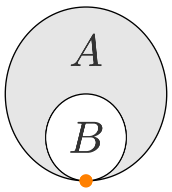
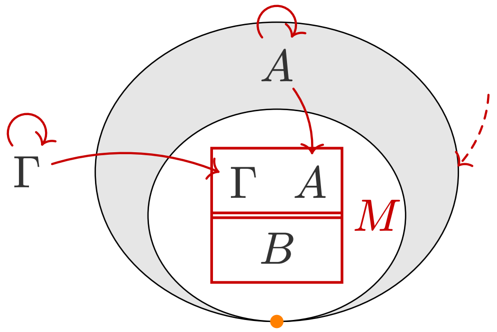
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
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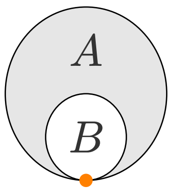
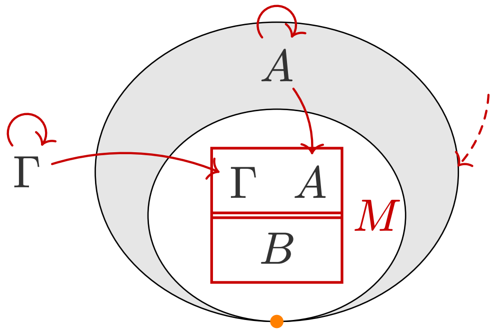
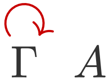
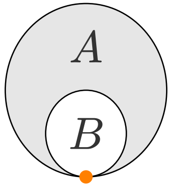
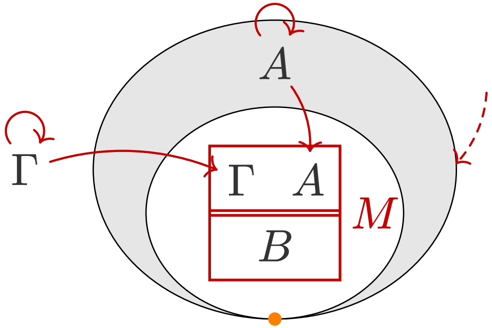
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
$\dfrac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)N : B}$	Γ	Γ

Completeness

From every ND proof of $\Gamma \vdash A$, build a **correct** scroll net S with $\lceil S \rceil = \Gamma$ and $\lfloor S \rfloor = A$:

$\dfrac{}{\Gamma, x : A \vdash x : A}$	A	
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
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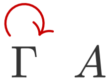
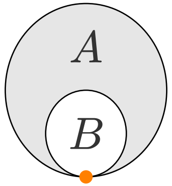
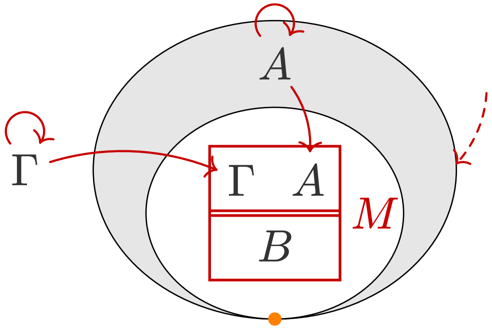
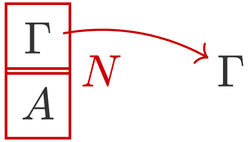
$$\frac{}{\Gamma, x : A \vdash x : A}$$	A	
$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$$		
$$\frac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)N : B}$$	A $\qquad$ Γ	

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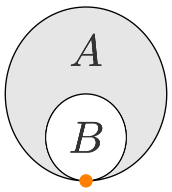
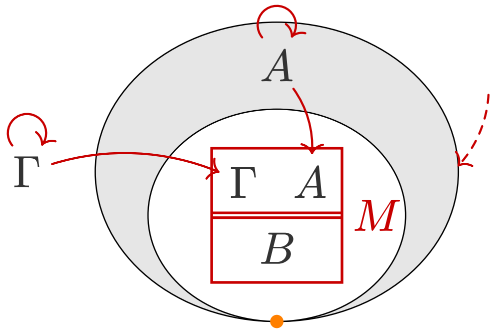
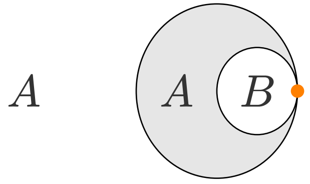
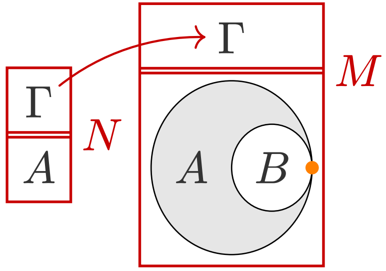
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
$\dfrac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)N : B}$		

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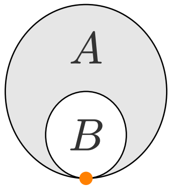
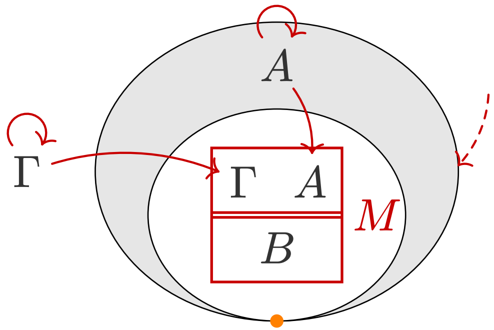
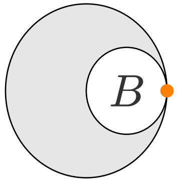
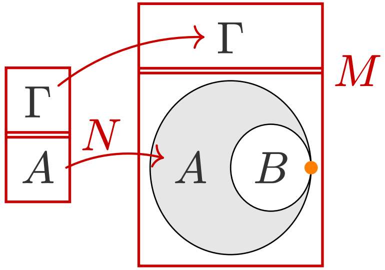
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
$\dfrac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)N : B}$	A 	

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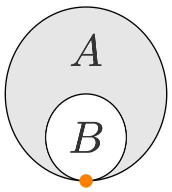
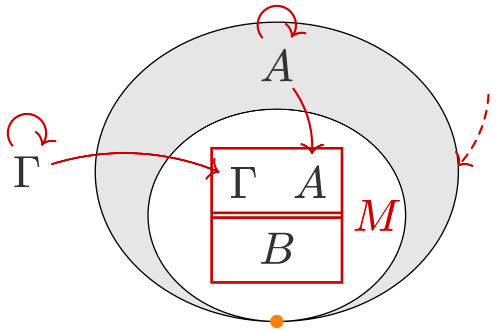
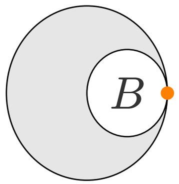
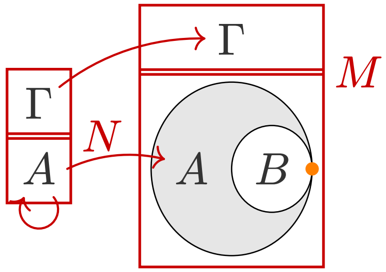
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
$\dfrac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)N : B}$		

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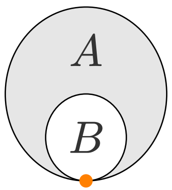
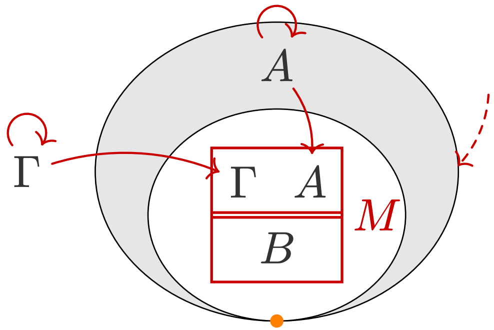
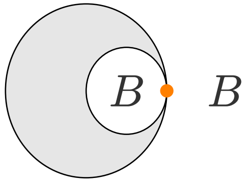
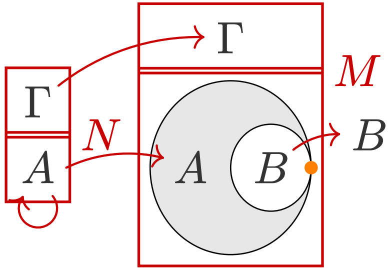
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
$\dfrac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)N : B}$		

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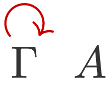
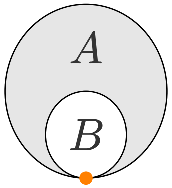
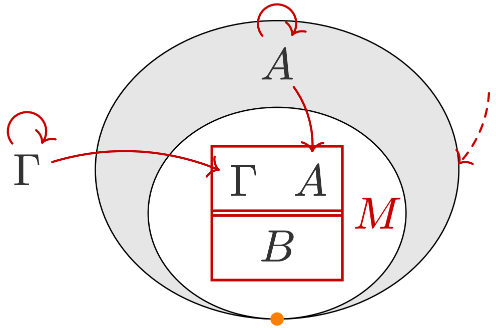
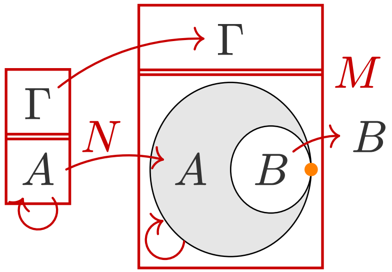
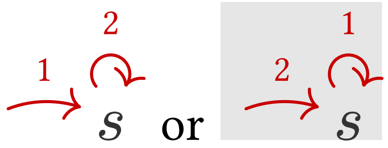
$\dfrac{}{\Gamma, x : A \vdash x : A}$	A	
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$		
$\dfrac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)N : B}$	B	

Detours

Detour $\stackrel{\text{def}}{=}$

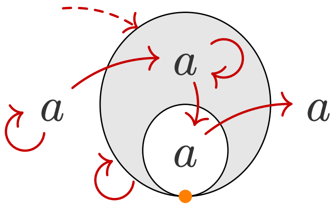


$y/\dot{x} : s$

	Introduction (1st)	**Elimination** (2nd)
Positive	Iteration	Deletion
Negative	Insertion	Deiteration

Definition 4 (Normal form) A scroll net is *normal* iff it contains no detour.

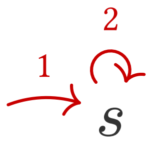
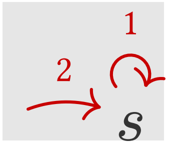
$(\lambda y.y)x$



$\dot{x} : a, \quad / \cdot [x/\dot{y} : a \; (y/z : a)], \quad z/a$

Detour reduction must preserve $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$
(*subject reduction*)

Detours

Detour $\stackrel{\text{def}}{=}$  or 

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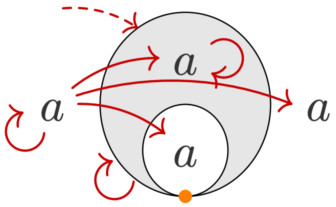

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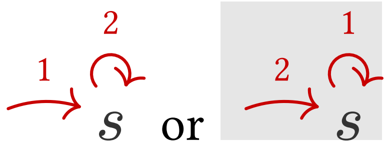
(*subject reduction*)

Detours

Detour $\stackrel{\text{def}}{=}$ 

$$y/\dot{x} : s$$

	Introduction (1st)	**Elimination** (2nd)
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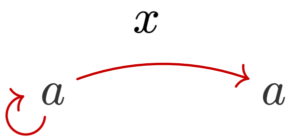
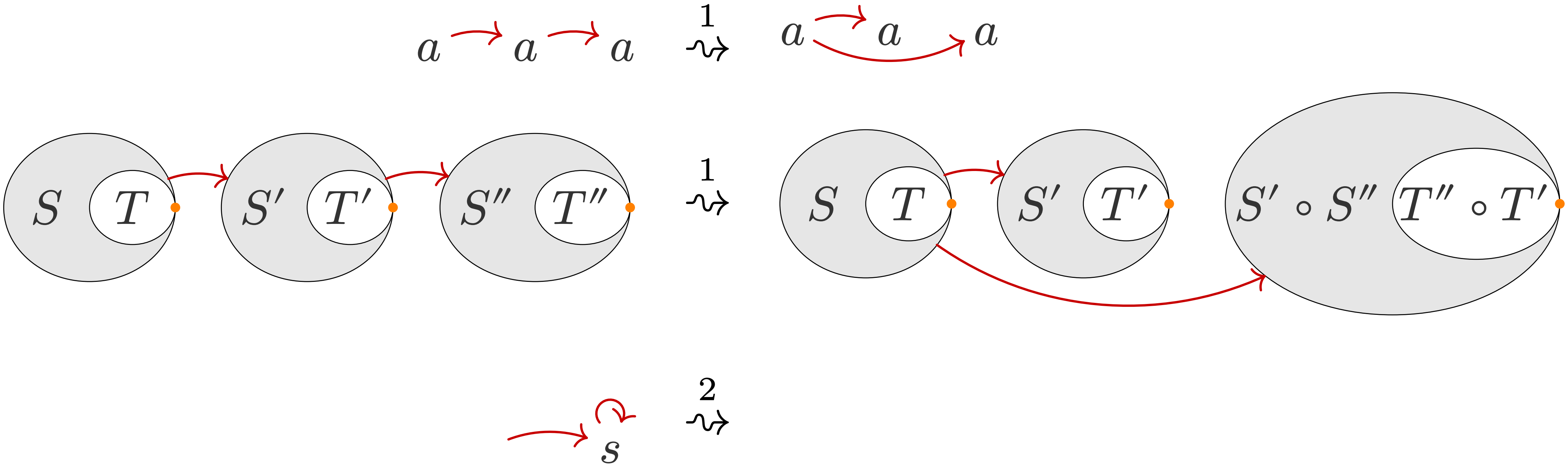


$$\dot{x} : a, \quad x/a$$

Detour reduction must preserve $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$
(*subject reduction*)

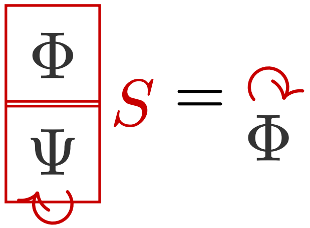
Operational semantics

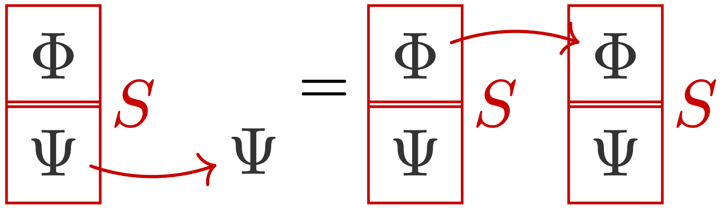
1. **(Rerouting)** Eliminate indirections through *tree rotations*
2. **(Garbage collection)** Remove unused detours



Denotational semantics

Inspired by the laws of **cocommutative comonoids** in *cartesian monoidal categories.*





$$\cdot\, \Psi \circ S = \cdot\, \Phi \qquad (\vec{x} : \Psi, \vec{x}/\Psi) \circ S = (S, S) \circ (\vec{x} : \Phi, \vec{x}/\Phi)$$

Composition (the major roadblock)

Intuitively:

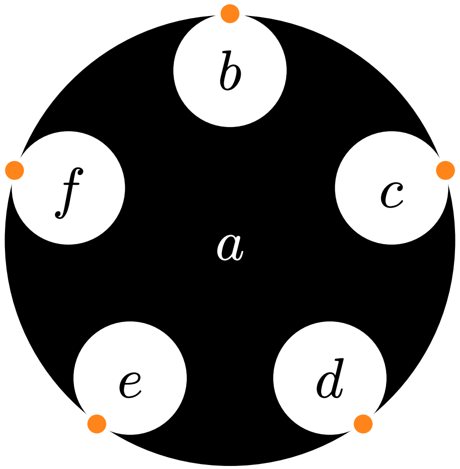
$$\begin{array}{|c|}\hline \Psi \\ \hline \Xi \\ \hline\end{array}\,T \circ \begin{array}{|c|}\hline \Phi \\ \hline \Psi \\ \hline\end{array}\,S = \begin{array}{|c|}\hline \Phi \\ \hline \Psi \\ \hline \Xi \\ \hline\end{array}\begin{array}{l} S \\ T \end{array}$$

Or, “execute” S to rewrite Φ into Ψ, then T to rewrite Ψ into Ξ.

Seems necessary to formalize the semantics…

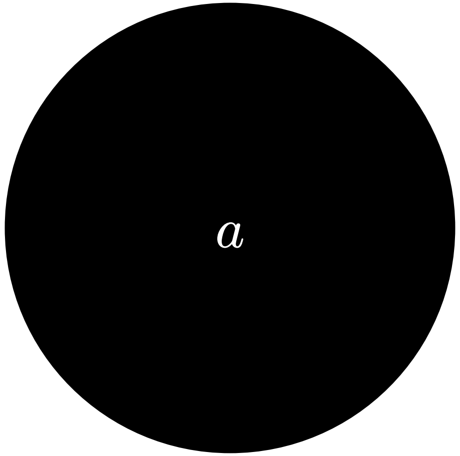
Conclusion

n-ary scroll (Oostra 2010)



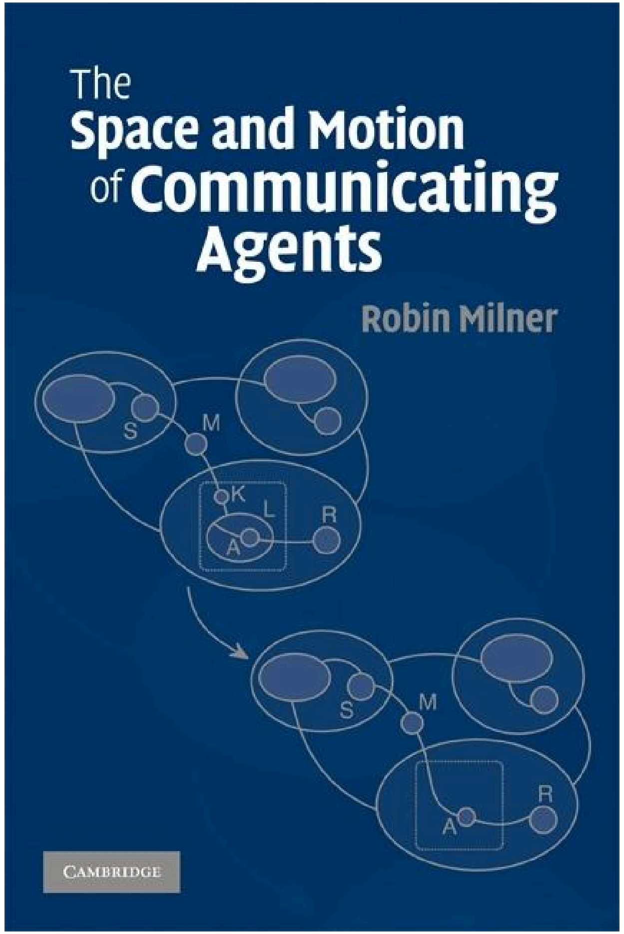
$$a \Rightarrow b \vee c \vee d \vee e \vee f$$

n-ary scroll (Oostra 2010)



$$\neg a \stackrel{\mathbf{def}}{=} a \Rightarrow \bot$$

Bigraphs (Milner 2009)



Place graph = Scroll structure

Link graph = Justification arrows

Bigraphs form a category

Homology of proof nets (Métayer 1994)

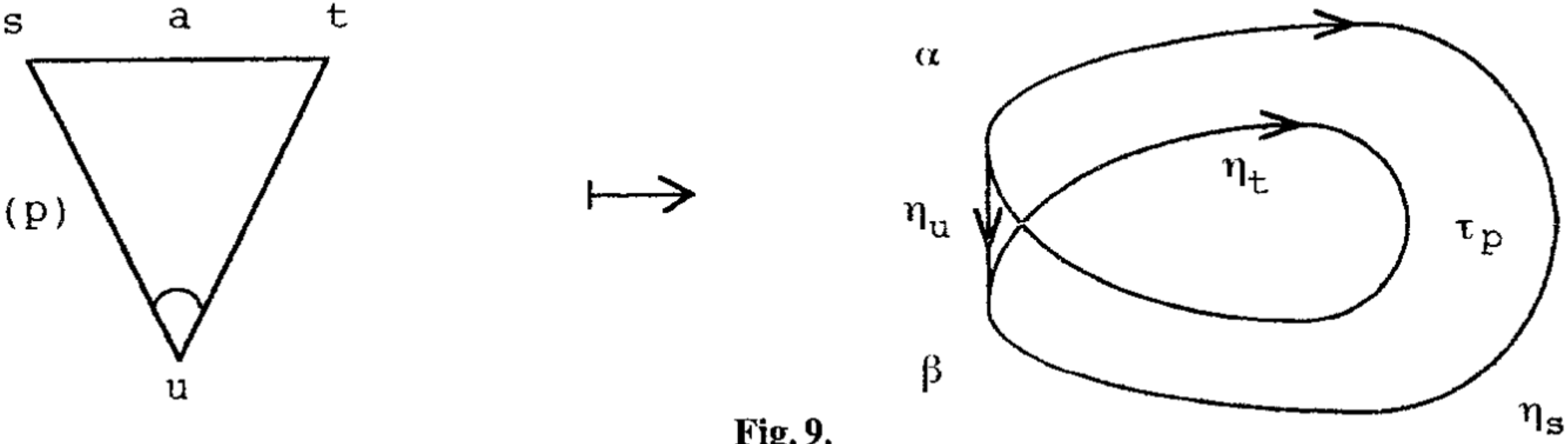


Fig. 9.

Summary

- A new syntax for proofs, *maybe* programs…
- …from an old system: faithful Curry-Howard reading of Peirce's EGs
- Lots of avenues for future research:
  - Analogue of proof-nets **correctness criterions** + sequentialization theorem
  - Formal **operational/denotational/categorical semantics** (ongoing)
  - Extensions to **richer logics** ($\{\vee, \bot\}$, classical/bi-intuitionistic logic, modal logics, linear logic, dependent types)
  - **Embedding of other proof/program formalisms** (sequent calculus/abstract machines, deep inference, π-calculus…?)
  - **Hilbert's 24th problem** (is it as good as *combinatorial proofs*?)
  - Connection to **Milner's bigraphs** (potential solution for composition?)
  - Connection to **homology**
  - …

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